JAMES RUSE AGRICULTURAL HIGH SCHOOL

TERM TWO 3 UNIT ASSESSMENT TASK

2000

INSTRUCTIONS:

1. Time allowed is 85 minutes
2. This is an open book test
3. Show all necessary formulae and working
4. Marks may be deducted for untidy or careless work
5. Start each section on a new sheet of paper
6. Approved calculators may be used

Section A (10 marks)

a) i) The motion of a particle moving in simple harmonic motion is given by $x = 3\sin(2t + \frac{\pi}{4})$, prove that $\ddot{x} = -4x$.

ii) Find the period of the motion

iii) When is it first at rest?

b) Melody buys two tickets in a raffle; Miles buys five in the same raffle. If 200 tickets were sold and there are three prizes, find the probability that:

i) Melody wins the first and second drawn prizes only.
ii) Melody wins only one prize and Miles does not win any.

c) A particle executing SHM moves such that $v^2 = 48 + 10x - 2x^2$, find the amplitude of the motion of the particle.

SECTION B (10 marks) Start a new page

a) If the acceleration a m/s^2 is given by $a = 6x + 1$ and initially $v = 2$ when $x = 1$, find the speed when $x = 2$.

b) An object is left lying in the sun on the beach. The rate of change of it's temperature is give by R deg/min, where $R = k(T - 16)$ and k is a constant.

i) Prove that $T = 16 + Ae^{kt}$ where A is a constant and t is the time in minutes, satisfies this condition.

ii) Initially the object has a temperature of 0°C and after 10 minutes it's temperature is 12°C, find the exact values of A and k.

iii) Find the temperature of the object after a further five minutes.
iv) Sketch a neat graph of the temperature against time, indicating its behaviour as $t \to \infty$

SECTION C (10 marks) Start a new page

a) Charissa, Aaron, John and Peter go to the Gold Coast for Schoolies Week. The Hotel has three spare rooms.

i) How many different ways can they be accommodated?
ii) What is the probability that two of the boys share a room if it is known that Charissa has a room to herself?

check b) If $v = \dfrac{4}{2t+3}$ m/s find:

i) An expression for x in terms of t, if initially the particle is at the origin.
ii) The initial acceleration
iii) Describe the motion of the particle.

SECTION D (10 marks) Start a new page

a) A particle is moving along a straight line, with velocity v m/s and acceleration given by th expression $k(4 - v^2)$ m/s^2 where k is a constant.

ok i) Show that $v^2 = 4 + Ae^{-2kx}$ where A is a constant satisfies the acceleration condition.
ii) If it is starts from $x = 0$ with a velocity of 7m/s, find the value of A.
iii) Does the particle ever change direction? Justify your answer.
iv) At $x = 1$, $v = 4$m/s, find the speed correct to two decimal places when $x = 2$.
v) As the motion continues, what happens to the velocity and the acceleration?

b) There is a Math's Association meeting in the James Ruse Theatrette with all members present. Merv is flying a Bomber overhead at an altitude of 3km at a speed of 250km/h. How far from the Theatrette should Merv release the bomb, in order to blow it up successfully? Take g = 10 m/s² and ignore air resistance.

SECTION E (10 marks) Start a new page

a) The letters of the word "logarithm" are arranged at random in a straight line. How many different arrangements are there if the vowels must be together?

b) Nine people go to a restaurant for dinner. They are given two circular tables; one of which seats six and the other three.

i) How many different seating arrangements around the two tables are possible?

ii) What is the probability that Mr and Mrs Canty find themselves at separate tables?

c) A substance contains two radioactive elements X and Y with half-lives of T_1 and T_2 respectively, and $T_1 > T_2$. Initially, the mass of Y is twice that of X. Prove that the substance will contain equal masses of X and Y after a time of $\dfrac{T_1T_2}{T_1 - T_2}$

SECTION F (10 marks) Start a new page

a) A projectile is fired from a point on horizontal ground with initial speed v m/s at an angle of elevation of α, where $\alpha \geq 45°$. It has a constant downward acceleration of gm/s².

i) Prove that the projectile will be climbing at an angle of elevation of 45° after a time of $\dfrac{v(\sin\alpha - \cos\alpha)}{g}$ seconds.

ii) If this time is $\dfrac{1}{3}$ of the total time of flight, find the value of $\tan\alpha$.

b) In a certain harbour at 6am it is low tide and at 12:30pm it is high tide. Low tide is at 4.5 metres and high tide is 10.6 metres. Assume that the motion of the tide can be approximated by SHM.

i) Write down an equation to represent the depth of water at time, t.

ii) If a ship requires a minimum depth of 8 metres, when can it first enter the harbour?

after 6am.

THE END

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2000 3UNIT SOLUTIONS

SECTION A (10marks)

1) i) $x = 3\sin(2t + \pi/4)$

$\dot{x} = 6\cos(2t + \pi/4)$

$\ddot{x} = -12\sin(2t + \pi/4)$

$= -4 \times 3\sin(2t + \pi/4)$

$\therefore \ddot{x} = -4x$

ii) period is $\frac{2\pi}{n} = \frac{2\pi}{2} = \pi$

iii) at rest when $\dot{x} = 0$

ie $\cos(2t + \pi/4) = 0$

$2t + \pi/4 = \pi/2$

$t = \pi/8$

$\therefore$ first at rest when $t = \pi/8$ seconds

2) i) $(W,W,L) = \frac{2}{200} \times \frac{1}{199}$

$= \frac{1}{19900}$

ii) $3(W,L,L) = 3 \times \frac{2}{200} \times \frac{193}{199} \times \frac{192}{198}$

$= \frac{1544}{54725}$

c) $v^2 = 48 + 10x - 2x^2$

at rest at endpoints, $v = 0$

$0 = x^2 - 5x - 24$

$0 = (x-8)(x+3)$

$x = 8$ or -3

$\therefore$ amplitude is $5\frac{1}{2}$.

SECTION B (10marks)

a) $a = 6x + 1$

$\int \frac{d(\frac{1}{2}v^2)}{dx}\,dx = \int (6x+1)\,dx$

$\frac{1}{2}v^2 = 3x^2 + x + c$

$t = 0, v = 2, x = 1$

$\therefore 2 = 3 + 1 + c$

$c = -2$

$\frac{1}{2}v^2 = 3x^2 + x - 2$

when $x = 2$,

$\frac{1}{2}v^2 = 3 \times 4 + 2 - 2$

$v^2 = 24$

$v = \pm\sqrt{24}$

$\therefore$ speed is $2\sqrt{6}$ m/s.

b) i) $T = 16 + Ae^{kt}$

$\frac{dT}{dt} = kAe^{kt}$

$\frac{dT}{dt} = k(T - 16)$

$\therefore T$ satisfies this condition

ii) $t = 0, T = 0°C$

$0 = 16 + Ae^0$

$A = -16$

$12 = 16 - 16e^{10k}$

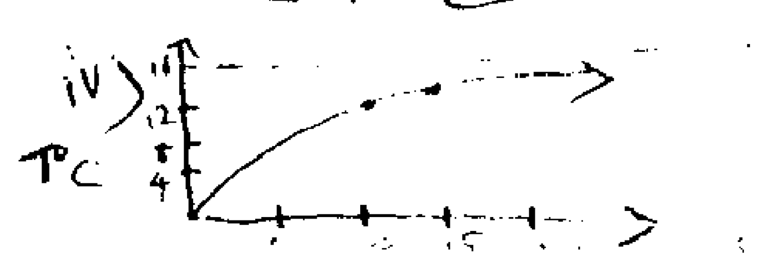
$\therefore e^{10k} = \frac{1}{4}$ $\qquad \frac{\ln 0.25}{10} = k$

iii) $T = 16 - 16e^{15k}$

$= 16 - 16e^{\frac{15\ln 0.25}{10}}$

$= 14°C$

iv)



SECTION C (10 marks)

a) i) 4 in one room

3 in one room, 1 in one

2 in one room, 2 in one

2 in one room, 1 in, 1 in

no. of arrangements is $\therefore$

$3 + (4 \times 3 \times 2) + \left(\frac{^4C_2 \times 3 \times {}^2C_2 \times 2}{2!}\right) + (^4C_2 \times 3 \times 2 \times 1)$

$= 3 + 24 + 18 + 36$

$= 81$

ii) Prob. $= \frac{^3C_2 \times 3 \times 2}{(3 \times 2) + (^3C_2 \times 3 \times 2 \times 1)}$

$= \frac{18}{24} = \frac{3}{4}$

b) i) $v = \frac{4}{2t+3}$

$x = 2\ln(2t+3) + c$

$t = 0, x = 0$

$0 = 2\ln 3 + c$

$\therefore x = 2\ln(2t+3) - 2\ln 3$

ii) $a = \frac{-8}{(2t+3)^2}$

when $t = 0$, $a = \frac{-8}{3^2}$

$= -\frac{8}{9}$ m/s^2

iii) Starts at the origin, moving to the right at $4/3$ m/s, but the acceleration is acting against the particle slowing it down, but the particle never actually stops.

SECTION D (10 mark

a) $a = k(4 - v^2)$

i) $v^2 = 4 + Ae^{-2kx}$

$\frac{1}{2}v^2 = 2 + \frac{A}{2}e^{-2kx}$

$a = \frac{d(\frac{1}{2}v^2)}{dx} = \frac{d(2 + \frac{A}{2}e^{-2kx})}{dx}$

$a = -2k \cdot \frac{A}{2}e^{-2kx}$

$a = -kAe^{-2kx}$

$a = -k(v^2 - 4)$

$\therefore a = k(4 - v^2)$

Q.E.D.

ii) $v = 7, x = 0$

$49 = 4 + Ae^0$

$\therefore A = 45$

iii) $v^2 = 0$ $\qquad 0 = 4 + 45$

but $e^{-2kx} \neq 0$

$\therefore v^2 \neq 0$

$\therefore$ never changes direc

iv) $v^2 = 4 + 45e^{-2kx}$

$x = 1, v = 4$

$16 = 4 + 45e^{-2k}$

$\ln(\frac{12}{45}) = -2k$

$k = -\frac{\ln 4/15}{2}$

when $x = 2$, $v^2 = 4 + 45$

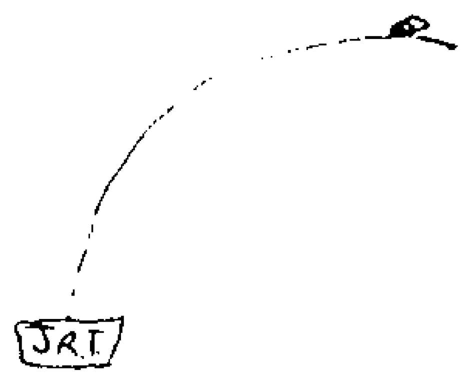
$v^2 = 7.2$

$v = 2.6833$

$\therefore$ speed is 2.68 m/

v) as $x \to \infty$ $\qquad v \to 2$

$a \to 0$



$x = 0$

$\dot{x} = 250 km/h$
$= 69.4 m/s$

$\dot{x} = 69.4t$

(1)

$\theta = 0^\circ,\ \dot{y} = 0$

$t = 0,\ y = 3km$

$\ddot{y} = -g$

$\dot{y} = -gt + c$

$\dot{y} = -10t$

$y = -5t^2 + 3000$

when $y = 0$, $0 = -5t^2 + 3000$

$\frac{3000}{5} = t^2$

(1) $t = \sqrt{600}$ (as $t > 0$)

$\therefore x = 69.4 \times \sqrt{600}$

(1)

$\therefore$ when Merv is 1701 m away from it!!

SECTION E (10 marks)

LOGARITHM

a) no. of arrangements $= 7!\,3!$

b) i) ${}^{9}C_6 \times 5! \times {}^{3}C_3 \times 2!$

$= 20160$

ii) pb. $= \frac{2 \times C_5 \times 5! \times 2!}{20160} = \frac{10080}{20160}$

$= \frac{1}{2}$

c) let mass of x be $x = Ae^{-kt}$

when $x = \frac{1}{2}A$ $t = T_1$

$\therefore \frac{1}{2} = e^{-kT_1}$

$k = \frac{\ln 2}{T_1}$

$\therefore x = Ae^{-\frac{\ln 2}{T_1}t}$

$\therefore y = 2Ae^{-\frac{\ln 2}{T_2}t}$

when $x = y$,

$Ae^{-\frac{\ln 2}{T_1}t} = 2Ae^{-\frac{\ln 2}{T_2}t}$

$e^{-\frac{\ln 2}{T_1}t} = 2e^{-\frac{\ln 2}{T_2}t}$

$\frac{e^{-\frac{\ln 2}{T_1}t}}{e^{-\frac{\ln 2}{T_2}t}} = 2$

$e^{\left(-\frac{\ln 2}{T_1}t + \frac{\ln 2}{T_2}t\right)} = 2$

$\ln 2 = -\frac{\ln 2}{T_1}t + \frac{\ln 2}{T_2}t$

$\ln 2 = t\left(\frac{\ln 2}{T_2} - \frac{\ln 2}{T_1}\right)$

$\ln 2 = t\ln 2\left(\frac{1}{T_2} - \frac{1}{T_1}\right)$

$1 = t\left(\frac{1}{T_2} - \frac{1}{T_1}\right)$

$\frac{1}{\frac{1}{T_2} - \frac{1}{T_1}} = t$

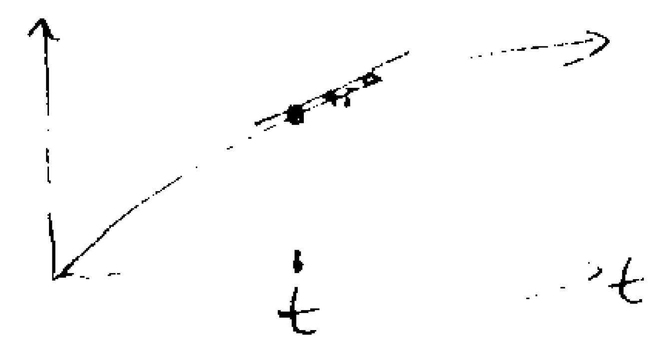
$t = \frac{1}{\frac{T_1 - T_2}{T_1T_2}}$

$\therefore t = \frac{T_1T_2}{T_1 - T_2}$

Q.E.D.

SECTION F (10 marks)

a)



$\ddot{x} = 0$

$\dot{x} = v\cos\alpha$

$x = vt\cos\alpha$

$\ddot{y} = -g$

$\dot{y} = -gt + v\sin\alpha$

$y = -\frac{g}{2}t^2 + vt\sin\alpha$

i) $\tan 45^\circ = 1$ (1)

$\frac{\dot{x}}{\dot{y}} = 1$

$\frac{v\cos\alpha}{-gt + v\sin\alpha} = 1$ (1)

$v\cos\alpha = -gt + v\sin\alpha$

$\frac{v(\cos\alpha - \sin\alpha)}{-g} = t$

$\therefore t = \frac{v(\sin\alpha - \cos\alpha)}{g}$ (1)

ii) $y = 0$ $0 = t\left(-\frac{g}{2}t + v\sin\alpha\right)$

(1) $t = 0$ or $t = \frac{2v\sin\alpha}{g}$

$\therefore \frac{v(\sin\alpha - \cos\alpha)}{g} = \frac{2v\sin\alpha}{3g}$

(1) $\sin\alpha - \cos\alpha = \frac{2\sin\alpha}{3}$

$3\sin\alpha - 3\cos\alpha = 2\sin\alpha$

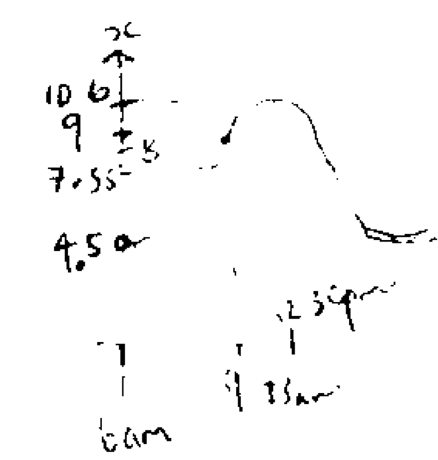
$\sin\alpha - 3\cos\alpha = 0$

(1) $\therefore \tan\alpha = 3$

b) period is 13 hours

$13 = \frac{2\pi}{n}$

$n = \frac{2\pi}{13}$ (1)



amp =

i) $x = 7.55 - 3.05\cos($

ii) $8 = 7.55 - 3.05\cos($

$\frac{0.45}{-3.05} = \cos\left(\frac{2\pi}{13}t\right)$

$t = \frac{13}{2\pi}\cos^{-1}\left(\frac{-0.45}{3.05}\right)$

$t = 3.556\,38290$

$= 3$ hrs 33 mins

$\therefore$ first enters harbour at