YEAR 12 ASSESSMENT

TERM 2 2004

MATHEMATICS
Extension I

INSTRUCTIONS:

Time allowed – 85 minutes plus 5 minutes reading time

All questions may be attempted

All questions are of equal value

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

The examination is an open book test

Standard integral tables will be supplied.
Approved templates and silent calculators may be used.

The answers to all questions are to be returned in separate bundles, each clearly labelled with the Question or Section number. Each bundle must show your candidate number

Question 1 (9 Marks) **Marks**

(a) How many seven letter words can be formed using the letters of the word "PRESSES". 2

(b) The velocity – time graph of a particle moving in a straight line is shown.
The velocity, $v = f(t)$, is in metres/second and the time, $0 \le t \le 4$, in seconds.

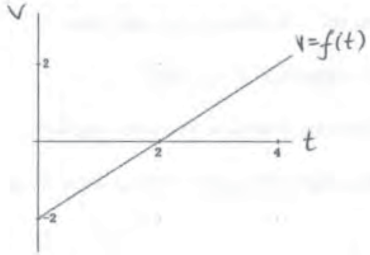


Diagram not to scale.

(i) Find the displacement, x metres, of the particle in the 2^{nd} second. 2

(ii) Find the acceleration of the particle at any time t. 1

(c) The acceleration of a particle moving in a straight line is given by $a = e^{-x+1}$, 4
where x is the displacement of the particle from the origin, O.

Initially, the particle starts at the origin with velocity 1 metres/second. Find $v(x)$.

Question 2 (9 Marks) **Start a new page.**

A hot cup of coffee loses heat in a colder environment according to Newton's Law of Cooling, $\frac{dT}{dt} = -k(T - T_e)$ at time t minutes. T_e is the temperature of the Environment and k is a constant.

(i) Show that $T = T_e + Ae^{-kt}$ is a solution of this equation, for any constant A. 2

(ii) I make my cup of coffee and find that the temperature is 90^0Celcius. 1
If the temperature of the room is 20^0Celcius, find the value of A.

(iii) The coffee then cools to 50^0Celcius after 6 minutes. Find the exact value of k. 2

(iv) Find how long it takes to reach 30^0Celcius, to the nearest second. 2

(v) Draw a neat sketch of the function, $T(t)$, for any time $t \ge 0$. 2

Question 3 (9 Marks) **Start a new page.** **Marks**

(a) There are 10 points on a 2 dimensional plane, no three of which are collinear. 5 are lettered A, B, C, D, E and 5 are lettered α, β, γ, δ, ε. Using these points, find how many triangles are possible with

(i) no restrictions. 1

(ii) 2 vertices denoted by capital letters and 1 denoted by a Greek letter. 1

(iii) Find the probability that a triangle formed has α as a vertex. 1

(iv) I) Using the same 10 points, find how many straight lines are possible? 1

II) If the 10 points form a dodecagon, find the number of diagonals formed. 1

(b) The acceleration of a particle moving along a straight line is given by $\ddot{x} = \cos^2 t - \sin^2 t + 1$.

(i) Find the velocity of the particle given that at $t = \dfrac{\pi}{2}$ seconds, $v = \dfrac{\pi}{2}$ metres/ second. 3

(ii) Describe the motion of the particle. 1

Question 4 (9 Marks) **Start a new page.**

(a) If ${}^5P_r = 20$, show that there is only one possible solution for r. 2

(b) The velocity, v metres/second, of a particle is given by $v^2 = 9 + 7x - 2x^2$.

(i) Show that the motion is simple harmonic. 3

(ii) Between which 2 points is the particle oscillating? 2

(iii) What is the speed of the particle as it moves through the equilibrium position? 2

Question 5 (9 Marks) **Start a new page.** **Marks**

(a) There are 3 different Red, 4 different Blue and 2 different Yellow keys on a table. Five different keys are chosen at random.

(i) How many ways can I place any 5 keys in a circle on a table? 2

(ii) How many ways can I choose exactly 2 Red keys? 2

(iii) Find the probability that in a circle of keys, I have exactly 2 Red keys. 1

(b) A particle is projected upwards with a velocity of 30 m/s at the point O. It comes back to the ground 50 metres away from the initial position. 4

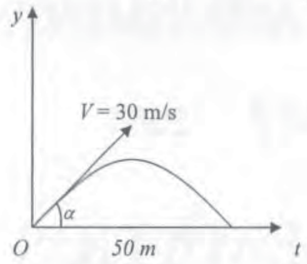


Diagram not to scale

Find the angle, α, that is required for the particle to travel this distance. Take $g = 10$ m/s^2.

Question 6 (9 Marks) **Start a new page.**

(a) A particle has displacement given by the function $x(t) = 2\log_e(\cos(t))$, where x is in centimetres and t is in seconds.

(i) Find the velocity after $\dfrac{\pi}{4}$ seconds. 2

(ii) Find the exact time when the particle will first be $\log_e\left(\dfrac{3}{4}\right)$ centimetres to the right of the origin. 2

(b) On a certain day the depth of water in a harbour at low tide at 12 noon is 4 metres. At the following high tide the depth of water is 11 metres and the interval between successive high tides is 12 hours 20 minutes. Assuming the rise and fall of the surface of the water to be simple harmonic, find the earliest time after 12 noon that a ship may safely enter the harbour if the minimum depth of 9 metres of water is required. 5

Question 7 (9 Marks) **Start a new page.**

(a) A particle moving in simple harmonic motion, has displacement, y metres, from the origin such that $y = \sin\left(2t - \frac{\pi}{4}\right) + \sin(2t)$. 4

Find the amplitude of this motion.

(b) A projectile is fired from a point O with a velocity of 2 metres per second, with an angle of elevation α, from the horizontal. The ground on which the projectile is sited is inclined at an angle of 30^0 to the horizontal. The projectile lands on the inclined ground at P, at a distance R metres from O.

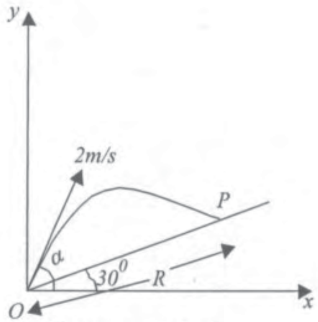


Diagram not to scale

Prove that:

(i) The time, T, which elapses before the projectile reaches the ground at a distance, R metres along the surface of the ground is given by $T = \frac{\sqrt{3}R}{4\cos\alpha}$. 2

(ii) The range, R metres, along the surface is given by $R = \frac{8\cos\alpha\sin\left(\alpha - 30^0\right)}{g\cos^2 30^0}$. 3

[Note: You do not need to prove the equations of projectile motion].

~ End of the Examination ~

Suggested Solutions ~ Mathematics E1 Term 2, 2004.

Question 1 (9 Marks) — Marks

(a) PRESSES has 7 letters, 3 S's and 2 E's. $\therefore$ no. of words $= \dfrac{7!}{2!3!} = 420$ ways. [2]

(b) (i) Displacement = area under velocity curve between $t = 1$ and $t = 2$.
$\therefore x = 0.5 \times 1 \times 1 = 0.5$ metres to the left as velocity is negative. [2]

Alternatively, the equation of the line $v = t - 2$ can be found.

$$\therefore \int_1^2 (t-2)d = \left.\frac{t^2}{2} - 2t\right]_1^2 = (2-4) - \left(\frac{1}{2} - 2\right) = -\tfrac{1}{2}$$

$\therefore$ displacement is ½ metres to the left of the Origin.

(ii) Acceleration = gradient function of $v(t) = 1$ [1]
$\therefore$ acceleration is 1 ms^{-2} to the right of the Origin.

(c) $a = e^{-x+1}$ $\quad \therefore v^2 = 2\int e^{-x+1}dx$ [1]

$= -2e^{-x+1} + c$ [1]

initially, $x = 0$, $v = 1$ $\therefore c = 1 + 2e$. [1]

$\therefore v = \sqrt{-2e^{-x+1} + 2e + 1}$ since $a > 0$, $v > 0$ [1]

Question 2 (9 Marks)

(i) $T = T_e + Ae^{-kt}$ $\therefore \dfrac{dT}{dt} = -kAe^{-kt}$ but $Ae^{-kt} = T - T_e$ $\therefore \dfrac{dT}{dt} = -k(T - T_e)$ [2]

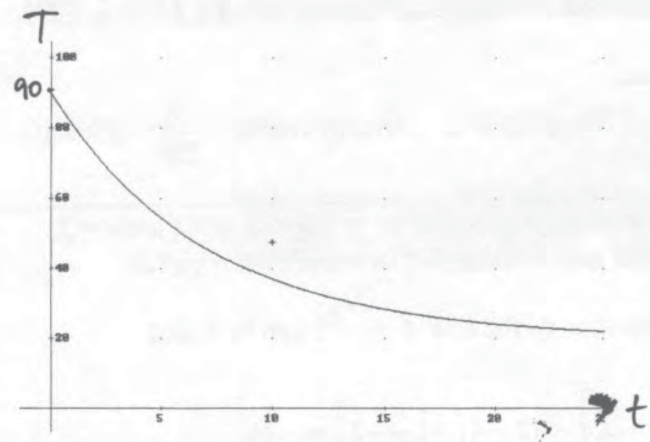
(ii) $t = 0$, $T = 90$ & $T_e = 20$ $\therefore A = 70^0$Celcius. [1]

(iii) $t = 6, T = 50$ $\therefore 30 = 70e^{-6k} \Rightarrow k = -\dfrac{1}{6}\ln\dfrac{3}{7}$ [2]

OR $k = \dfrac{\ln 7 - \ln 3}{6}$

(iv) If $T = 30$ $\quad \therefore t = \dfrac{6\ln\frac{1}{7}}{\ln\frac{3}{7}} =$ 13 mins 47 seconds [2]

(v) [2]



Question 3 (9 Marks) **Marks**

(a) (i) No. of triangle with no restrictions = $^{10}C_3$ = 120 [1]

(ii) 2 vertices with capitals & 1 Greek = $^5C_2 \times {}^5C_1$ = 50 [1]

(iii) If α is fixed the 2 other vertices can only be chosen $^9C_2$. = 36

$\therefore$ Probability = $\dfrac{^9C_2}{^{10}C_3} = \dfrac{3}{10}$ [1]

(iv) I) No. of straight lines = $^{10}C_2$ = 45 [1]

II) No. of diagonals = $^{10}C_2 - 10$ = 35 [1]
(since 10 if the straight lines are the sides of the dodecagon)

(b) (i) $\ddot{x} = \cos^2 t - \sin^2 t + 1$

$\therefore\ v = \int(\cos(2t)+1)dt$ [1]

$= \dfrac{\sin(2t)}{2} + t + k$ [1]

When $t = \dfrac{\pi}{2}, v = \dfrac{\pi}{2}$ $\quad\therefore\ k = 0$ [1]

$\therefore\ v = \dfrac{\sin(2t)}{2} + t$

(ii) [1]

Particle starts at rest ($v = 0$) with $a = 2$ m/s^2 . Moves to right indefinitely.

Question 4 (9 Marks)

(a) $^5P_r = 20$ i.e. $\dfrac{5!}{(5-r)!} = 20$ $\quad\therefore\ (5-r)! = 6 = 3!$ $\quad\therefore\ r = 2$ only. [2]

(b) (i) $v^2 = 9 + 7x - 2x^2$

$$= 2\left(\frac{9}{2} + \frac{7x}{2} - x^2 + \left(\frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2\right) = 2\left[\left(\frac{7}{4} - x\right)^2 - \frac{121}{16}\right] \quad [2]$$

which is of the form $v^2 = n^2(x^2 - a^2)$ $\therefore$ motion is SHM. [1]

(b) (ii) End points occur when $v = 0$

$\therefore (9 - 2x)(1 + x) = 0$ [1]

$\therefore x = 4.5$ or $x = -1$

$\therefore$ particle oscillates between $-1 \le x \le 4.5$ [1]

(iii) Equilibrium position, $x = \dfrac{7}{4}$ [1]

$\therefore$ speed $= |v| = \sqrt{\dfrac{121}{8}} = 3.889$ m/s [1]

Question 5 (9 Marks)

(a) (i) There are ${}^9C_5 \times 4! = 3024$ ways of choosing any 5 keys and putting them on a table. [2]

(ii) Exactly 2 red keys $= {}^3C_2 \times {}^6C_3 = 60$ ways [2]

(iii) $\therefore$ Probability is $\dfrac{{}^3C_2 \times {}^6C_3 \times 4!}{{}^9C_5 \times 4!} = \dfrac{10}{21}$ [1]

(b) The displacement of the particle is given by the equations $x = 30t\cos\alpha$ and

$y = -5t^2 + 30t\sin\alpha$

$\therefore t = \dfrac{5}{3\cos\alpha}$ and you get the Cartesian equation of $\dfrac{-125}{9\cos^2\alpha} + \dfrac{50\sin\alpha}{\cos\alpha} = 0$ [1]

$\therefore 5\tan^2\alpha - 18\tan\alpha + 5 = 0$ [1]

$\therefore \tan\alpha = 3.296$ or 0.303 by quadratic formula [1]

$\therefore \alpha = 16^0 52'$ or $73^0 8'$ [1]

Question 6 (9 Marks)

(a) (i) $x(t) = 2\log_e(\cos(t))$ $\quad \therefore v(t) = \dfrac{-2\sin(t)}{\cos(t)} = -2\tan(t)$ [1]

when $t = \dfrac{\pi}{4}$ then $v = -2$ m/s i.e. velocity is 2 m/s to the left of the origin. [1]

(ii) $\log_e\left(\frac{3}{4}\right) = 2\log_e(\cos(t))$ $\therefore \frac{3}{4} = \cos(t)^2$ [1]

$\therefore \cos(t) = \frac{\sqrt{3}}{2}$ (only since $x > 0$ initially) [1]

$\therefore t = \frac{\pi}{6}$. i.e. at $\frac{\pi}{6}$ seconds, the particle will first be at $\log_e(\cos(t))$ cms.

(b) $T = \frac{2\pi}{n}$, T = 12 hours 20 minutes. $\therefore n = \frac{2\pi}{12\frac{1}{3}} = \frac{6\pi}{37}$ [1]

amplitude = 3.5 metres , centre of motion = 7.5 metres.

$\therefore$ equation is $x = 7.5 - 3.5\cos\left(\frac{6\pi}{37}t\right)$ [1]

$\therefore$ when $x = 9 \rightarrow$ $\cos\left(\frac{6\pi}{37}t\right) = -\frac{1.5}{3.5}$. [1]

$\therefore t = 237$ minutes = 3 hours 57' [1]

i.e. the boat can enter the harbour at 3.57 p.m. [1]

Question 7 (9 Marks)

(a) $y = \sin\left(2t - \frac{\pi}{4}\right) + \sin(2t) = \sin(2t)\cos\frac{\pi}{4} + \cos(2t)\sin\frac{\pi}{4} + \sin(2t)$ [1]

i.e. $y = \frac{\sin(2t)}{\sqrt{2}} + \frac{\cos(2t)}{\sqrt{2}} + \sin(2t) = \sin(2t)\left[1 + \frac{1}{\sqrt{2}}\right] + \frac{\cos(2t)}{\sqrt{2}}$ [1]

y can be written in the form of $A\sin(2t + \beta)$, where A is the amplitude. [1]

$\therefore A = \sqrt{\frac{1}{2} + \left[1 + \frac{1}{\sqrt{2}}\right]^2} = \sqrt{2+\sqrt{2}}$ [1]

(b) (i) $T = \frac{\textit{horizontal displacement}}{\textit{horizontal velocity}} = \frac{R\cos 30^0}{2\cos\alpha} = \frac{R \times \frac{\sqrt{3}}{2}}{2\cos\alpha} = \frac{\sqrt{3}R}{4\cos\alpha}$ [2]

(ii) The Cartesian eqn. of the path travelled by the particle is given by

$y = \frac{-gx^2}{4\cos^2\alpha} + \frac{x\sin\alpha}{\cos\alpha}$ but $x = R\cos 30^0$ & $y = R\sin 30^0$

$\therefore R\sin 30^0 = \frac{-gR^2\cos^2 30^\circ}{4\cos^2\alpha} + \frac{R\cos 30^\circ \sin\alpha}{\cos\alpha}$ [1]

$\therefore \frac{\cos\alpha\sin 30^\circ - \sin\alpha\cos 30^\circ}{\cos\alpha} = \frac{-gR\cos^2 30^\circ}{4\cos^2\alpha}$ [1]

$\therefore \frac{-2\sin(\alpha - 30^\circ)}{\cos\alpha} = \frac{-gR\cos^2 30^\circ}{4\cos^2\alpha}$ [1]

$\therefore R = \frac{8\cos\alpha\sin(\alpha - 30^\circ)}{g\cos^2 30^0}$ as required.