Question 1 (10 marks) **Marks**

(a) The letters of the word CALCULUS are arranged in a row.

(i) How many different arrangements are possible ? **1**

(ii) If one of these arrangements is selected at random, what is the probability that it begins with a "U" and ends with a "U" ? **2**

(b) Liz heats a mug of milk up to $90°C$ in a microwave. She takes it out into a room where the temperature is a constant of $26°C$. The milk cools to $70°C$ in 5 minutes. At time t minutes, its temperature $T°$ decreases according to the equation

$$\frac{dT}{dt} = -k\,(T - 26), \quad \text{where } k \text{ is a positive constant.}$$

(i) Verify that $T = 26 + Ae^{-kt}$ is a solution of this equation, where A is a constant. **1**

(ii) Find the values of A and k. **2**

(iii) How long will it take for the temperature to cool to $30°C$? Give your answer to the nearest minute. **2**

(iv) Sketch the graph of T as a function of t. **2**

Question 2 (10 marks) Start a new page

(a) A particle P moves along a straight line so that at time t, its displacement from a fixed point 0 on that line is given by

$$x(t) = 3t^2(4 + t^3)^{-1}.$$

(i) Find an expression for the velocity of the particle at time t. **1**

(ii) Find the time when the particle is momentarily at rest after the motion has started. **1**

(iii) Show that P is in exactly the same position at both times $t_1 = 1$ and $t_2 = 2 + 2\sqrt{2}$. **2**

(iv) Graph the displacement – time function. **2**

(b) Two particles P and Q move along a line, their displacement at time t with respect to a fixed point O being $x(t)$ and $X(t)$ respectively.

(i) The acceleration of P is given by $\frac{d^2x}{dt^2} = 6 + e^{-t}$. If it begins its motion at $x=0$ with a velocity of -1, find an expression for $x(t)$. **2**

(ii) If $X(t)=2\sin 5t + 3t^2 + 2$, prove that $X(t) > x(t)$ for all $t \geq 0$. **2**

Question 3 (10 marks) Start a new page

Marks

(a) Prove that $\frac{d}{dx}(\frac{1}{2}v^2) = \frac{d^2x}{dt^2}$, where v is velocity and $\frac{d^2x}{dt^2}$ is acceleration as a function of time. **1**

(b) The acceleration of a particle moving in a straight line is given by $\frac{d^2x}{dt^2} = (4x - 4)$, where x is the displacement, in metres, from the origin O and t is the time in seconds. Initially the particle is 6 metres to the right of O and its velocity (v m/s) is -8 m/s.

(i) Show that $v^2 = 4x^2 - 8x - 32$. **2**

(ii) Find the set of possible values of x where motion can exist and describe the motion of the particle. **3**

(c) A particle is moving with simple harmonic motion in a straight line with a period of π seconds. Its maximum speed is 12 m/s. Initially the particle has a displacement of 3 metres from the centre of motion and is moving to the right.

If x is the displacement, in metres, from the centre of motion ($x=0$) and t is the time in seconds, find an expression for x in terms of t. **4**

Question 4 (10 marks) Start a new page

(a) A particle is moving in a straight line. At time t seconds its displacement x metres from a fixed point O on the line is given by

$$x = 2\cos^2 t.$$

(i) Prove that the motion is simple harmonic. **2**

(ii) Find the amplitude of the motion. **1**

(b) An employer wishes to choose two people for a job. There are eight applicants, three of whom are women and five of whom are men.

(i) If each applicant is interviewed separately and all of the women are interviewed before any of the men, find how many ways there are in carrying out the interviews. **1**

(ii) If the employer chooses two of the applicants at random, find the probability that at least one of those chosen is a woman. **2**

Marks

(c) A nine-member Fund Raising Committee consists of four students, three teachers and two parents. The Committee meets around a circular table.

(i) How many different arrangements of the nine members around the table are possible if the students sit together as a group, as do the teachers, but no teachers sit next to a student ? **2**

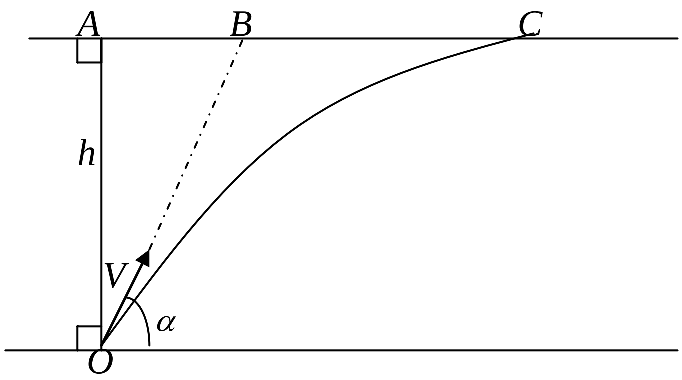
(ii) One student and one parent are related. Given that all arrangements in part (i) are equally likely, what is the probability that these two members sit next to each other ? **2**

Question 5 (10 marks) Start a new page

(a) Find how many groups of one or more digits can be formed from the following digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if repetition is not allowed. **2**

(b)

diagram not to scale



In the diagram, an aeroplane is flying with constant velocity U at a constant height h above horizontal ground.
When the plane is at A, it is directly over a gun at O.
When the plane is at B (time $t=0$), a shell is fired from the gun at the plane along the direction OB. The shell is fired with initial velocity V at an angle of elevation α.

The horizontal and vertical components of the displacement of the shell from O at time t are given respectively by

$$x = Vt\cos\alpha \quad \text{and} \quad y = Vt\sin\alpha - \frac{1}{2}gt^2,$$

while g is the acceleration due to gravity.

(i) Show that if the shell hits the plane at point C at time $t=T$, then **2**

$$VT\cos\alpha = \frac{h}{\tan\alpha} + UT.$$

(ii) Show that when the shell hits the plane then $2U(V\cos\alpha - U)\tan^2\alpha = gh$. **3**

Marks

(c) The velocity v of a particle at time t, is given in terms of its displacement, x, by the equation $v = \frac{4}{x}$, where $x \neq 0$. Initially, $x = 8$.

(i) Find an expression for the acceleration of the particle in terms of x. **1**

(ii) By expressing v as $\frac{dx}{dt}$, find an expression for x^2 in terms of t. **2**

Question 6 (10 marks) Start a new page

(a) Prove that ${}^{n+1}C_{k+1} = {}^{n}C_{k} + {}^{n}C_{k+1}$, for $1 \leq k < n$ and $n \geq 1$. **2**

(Do not use induction)

(b) The rate of change of the population of a country is affected by the maximum possible population M of the country. M depends on factors such as the area of land and the amount of raw materials etc.

If P is the population it can be shown that $\frac{dP}{dt} = kP(M - P)$,

where M and k are constants and t is measured in years,

(i) Verify that $P = \frac{AMe^{Mkt}}{Ae^{Mkt} + 1}$ is a solution to the equation where A is a constant. **2**

(ii) It is known that the maximum possible population (M) of a country is 860 million. In 1790 the population of the country was 4 million people and in 1800 the population was 6 million people. **5**

In what year was the population of the country equal to half of its maximum possible population (i.e. 430 million)? Give your answer to the nearest year.

(iii) Describe what happens to the population growth rate as P approaches M. **1**

END of PAPER

Solutions to 2005 T2 Yr 12 Ext 1.

Q1 a i) 2C 1A 2U 2L 1S Total 8 letters

$$\frac{8!}{2!\,2!\,2!} = \underline{\underline{5040}}$$

ii) No of arrangements begins with U and ends in U

$$= \frac{6!}{2!\,2!} = 180$$

$$\text{Prob} = \frac{180}{5040} = \underline{\underline{\frac{1}{28}}}$$

b) i) $T = 26 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$= -k(T-26)e^{-kt} \quad (\text{since } A = T - 26)$$

ii) $t=0$, $T = 90^\circ$

$$90 = 26 + A$$

$$\underline{A = 64}$$

$$\therefore T = 26 + 64e^{-kt}$$

when $t = 5$. $T = 70^\circ$

$$70 = 26 + 64e^{-5k}$$

$$\frac{44}{64} = e^{-5k}$$

$$-5k = \ln\left(\frac{44}{64}\right)$$

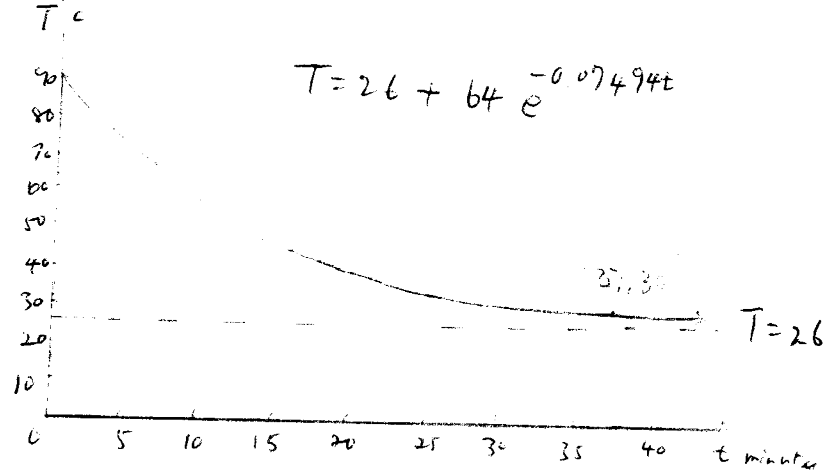
$$\underline{k = 0.07494 \ (4\text{ sf})}$$

iii) $30 = 26 + 64e^{-0.07494t}$

$$\frac{4}{64} = e^{-0.07494t}$$

$$\ln\left(\frac{1}{16}\right) = (-0.07494)t$$

$\underline{t = 37 \text{ min}}$ (nearest min) i b iv



Q 2

a) $x = \dfrac{3t^2}{4+t^3}$

i) $V = \dfrac{(4+t^3)\,6t - 3t^2(3t^2)}{(4+t^3)^2}$

$= \dfrac{24t + 6t^4 - 9t^4}{(4+t^3)^2}$

$= \dfrac{24t - 3t^4}{(4+t^3)^2}$

ii) $V = 0$ when $24t - 3t^4 = 0$

$8t - t^4 = 0$

$t(8 - t^3) = 0$

$t = 0$ or $t = 2$

$\therefore t = 2$

iii) $t_1 = 1, \quad x_1 = \dfrac{3}{4+1} = \dfrac{3}{5}$

$t_2 = 2 + 2\sqrt{2}, \quad x_2 = \dfrac{3(2+2\sqrt{2})^2}{4 + (2+2\sqrt{2})^3}$

$x_2 = \dfrac{3(4 + 8\sqrt{2} + 8)}{4 + 2^3 + 3(2^2 \cdot 2\sqrt{2}) + 3(2 \times 4 \cdot 2) + 8 \cdot 2\sqrt{2}}$

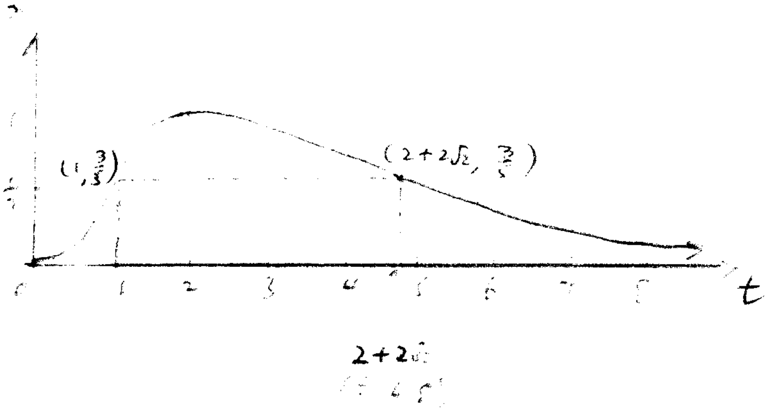
$= \dfrac{3(12 + 8\sqrt{2})}{12 + 24\sqrt{2} + 48 + 16\sqrt{2}}$

$= \dfrac{36 + 24\sqrt{2}}{60 + 40\sqrt{2}}$

At t_2 $\quad x_2 = \dfrac{12(3+2\sqrt{2})}{20(3+2\sqrt{2})} = \dfrac{3}{5}$

$\therefore x_1 = x_2$

iv)



2

2b i) $\ddot{x} = 6 + e^{-t}$ $\qquad \dot{x} = \int 6 + e^{-t}\,dt$

$\dot{x} = 6t - e^{-t} + k_1$

$t=0,\ \dot{x}=-1 \quad \therefore -1 = -e^{0} + k_1$

$\therefore k_1 = 0$

$\therefore \dot{x} = 6t - e^{-t}$ $\qquad$ 1

$x = \int 6t - e^{-t}\,dt$

$x = 3t^2 + e^{-t} + k_2$

$t=0,\ x=0,\quad 0 = 1 + k_2$

$-1 = k_2$

$\therefore \underline{\underline{x = 3t^2 + e^{-t} - 1}}$ $\qquad$ 1

ii) $X(t) - x(t) = 2\sin 5t + \cancel{3t^2} + 2 - (\cancel{3t^2} + e^{-t} - 1)$

$= 3 + 2\sin 5t - e^{-t}$

$\left.\begin{array}{l} \min \sin 5t = -1 \\ \min 2\sin 5t = -2 \end{array}\right\}\ 3 + 2\sin 5t \geq 1$ $\qquad \frac{1}{2}$

$\max e^{-t} = 1$ for $t \geq 0$ because e^{-t} is a decreasing function and $e^{0} = 1$ and $e^{-t} < 1$ for $t > 0$ $\qquad$ 1

At $t=0$ $\quad X(0) - x(0) = 3 + 0 - 1 = 2$ $\qquad \frac{1}{2}$

$\therefore X(t) - x(t) > 0$ for $t \geq 0$

Q 3

a) $$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx}$$

$$= v\frac{dv}{dx}$$

$$= \frac{dx}{dt} \cdot \frac{dv}{dx}$$

$$= \frac{dv}{dt}$$

$$= \frac{d^2x}{dt^2}$$

#

b i) $$\frac{d(\frac{1}{2}v^2)}{dx} = 4x - 4$$

$$v^2 = 2\int 4x - 4 \, dx$$

$$v^2 = 4x^2 - 8x + k$$

$t = 0$, $x = 6$, $v = -8$

$$64 = 4(36) - 48 + k$$

$$-32 = k$$

$$\therefore v^2 = 4x^2 - 8x - 32$$

ii) $$v^2 = 4(x-4)(x+2) \geq 0$$

$\therefore$ $x \geq 4$ or $x \leq -2$ for motion to exist

In this case initially $x = 6$ $v = -8$ $\ddot{x} = 20$

The particle starts at 6 m to the right of O, moving to the left, slowing down ($\ddot{x} = 20 > 0$) until it reaches 4 m to the right of O. It stops there momentarily, turn around and move to the right, speeding up forever and never return.

2

3c) Period $= \pi$

$\frac{2\pi}{n} = 2 \Rightarrow n = 2$ /

$\max |V| = 12$ at $x = 0$

$v^2 = n^2(a^2 - x^2)$

$144 = 4(a^2 - 0)$

$a = 6$ /

$t = 0$, $x = 3$ moving to the right

$x = 6\sin(2t + \alpha)$

$3 = 6\sin(\alpha)$

$\frac{1}{2} = \sin\alpha$

$\alpha = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ / for rejecting $\frac{5\pi}{6}$

But $\dot{x} = 12\cos(2t + \alpha)$

At $t = 0$, $\dot{x} = 12\cos\frac{\pi}{6} = 6$ (moving to the right)

$\dot{x} = 12\cos\frac{5\pi}{6} = -6$ (moving to the left)

$\therefore \alpha = \frac{\pi}{6}$

$\therefore x = 6\sin(2t + \frac{\pi}{6})$ /

Note: there are other alternative solutions

Q 4 ai) $x = 2\cos^2 t = 1 + \cos 2t$

$x - 1 = \cos 2t$

$\frac{dx}{dt} = -2\sin 2t$ /

$\frac{d^2x}{dt^2} = -4\cos 2t$

$= -2^2(x - 1)$ which is in the form of $\ddot{x} = -n^2(x - b)$ /

$\therefore$ SHM

ii) amplitude $= 1$ m /

Q4 b) i) $3!\,5! = 720$

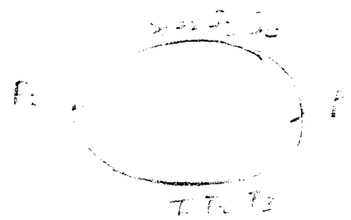
ii) $1 - P(\text{All men}) = 1 - \frac{5}{8} \times \frac{4}{7} = \frac{9}{14}$

c) i) Parents 2 ways

Students as a group $4!$

teachers as a group $3!$

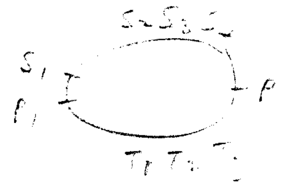
Total no. of ways $= 2 \times 4! \times 3! = 288$



ii) $S_1 P_1$ can swap with P

Total no of ways $= 2 \times 3! \times 3! = 72$

Prob $= \frac{72}{288} = \frac{1}{4}$



5 a) 10 different digits. Each can be in or out (2 choices)

but at least 1 digit needs to be included (ie exclude all digits out)

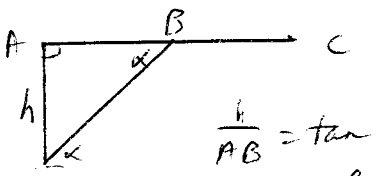
$2^{10} - 1 = \underline{\underline{1023}}$

b) $x = VT\cos\alpha$ horizontal distance travelled by shell from A to C

Horizontal distance travelled by plane from A to C

$= AB + BC$

$= \frac{h}{\tan\alpha} + uT$



From B C

Speed is u, constant

time is T

$\frac{h}{AB} = \tan\alpha$

$AB = \frac{h}{\tan\alpha}$

$\therefore \underline{\underline{VT\cos\alpha = \frac{h}{\tan\alpha} + uT}}$

5 biii) $$VT\cos\alpha = \frac{h}{\tan\alpha} + uT \quad \text{(from part ii)}$$

Solve for T : $$VT\cos\alpha - uT = \frac{h}{\tan\alpha}$$

$$T(V\cos\alpha - u) = \frac{h}{\tan\alpha}$$

$$T = \frac{h}{\tan\alpha\,(V\cos\alpha - u)}$$

$$y = Vt\sin\alpha - \tfrac{1}{2}gt^2 \quad \text{(from part i)}$$

At c, $t = T$, $y = h$

$$h = VT\sin\alpha - \tfrac{1}{2}gT^2$$

$$h = V\,\frac{h\sin\alpha}{\tan\alpha\,(V\cos\alpha - u)} - \frac{1}{2}g\,\frac{h^2}{(\tan\alpha)^2(V\cos\alpha - u)^2}$$

$$1 = \frac{V\cos\alpha}{(V\cos\alpha - u)} - \frac{1}{2}\cdot\frac{gh}{(\tan\alpha)^2(V\cos\alpha - u)^2}$$

$$1 = \frac{2(V\cos\alpha)(V\cos\alpha - u)(\tan\alpha)^2 - gh}{2(\tan\alpha)^2(V\cos\alpha - u)^2}$$

$$2(\tan\alpha)^2(V\cos\alpha - u)^2 - 2V\cos\alpha(V\cos\alpha - u)(\tan\alpha)^2 = -gh$$

$$2(V\cos\alpha - u)(\tan\alpha)^2\left[\cancel{V\cos\alpha} - u - \cancel{V\cos\alpha}\right] = -gh$$

$$-2u(V\cos\alpha - u)(\tan\alpha)^2 = -gh$$

$$\underline{2u(V\cos\alpha - u)(\tan\alpha)^2 = gh} \quad \text{QED.}$$

5 c) i) $\ddot{x} = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = \dfrac{d}{dx}\left[\frac{1}{2} \times \left(\frac{4}{x}\right)^2\right] = \dfrac{d}{dx}\left(\frac{8}{x^2}\right) = \dfrac{-16}{x^3}$

ii) $\dfrac{dx}{dt} = \dfrac{4}{x}$

$\int x\,dx = \int 4\,dt$

$\dfrac{x^2}{2} = 4t + k$

$t=0,\ x=8 \qquad \dfrac{64}{2} = k \quad \therefore k = 32$

$\therefore x^2 = 2(4t+32)$

$\underline{\underline{x^2 = 8t + 64}}$

6 a) $^{n}C_{k} + {}^{n}C_{k+1} = \dfrac{n!}{(n-k)!\,k!} + \dfrac{n!}{(n-k-1)!\,(k+1)!}$

$= \dfrac{n!\,(\cancel{k}+1 + n-\cancel{k})}{(n-k)!\,(k+1)!}$

$= \dfrac{(n+1)\,n!}{(n-k)!\,(k+1)!}$

$= \dfrac{n!}{(n-k)!\,(k+1)!}$

$= {}^{n+1}C_{k+1}$

b) $P = \dfrac{AMe^{Mkt}}{Ae^{Mkt}+1}$

$\dfrac{dP}{dt} = \dfrac{(Ae^{Mkt}+1)AM^2k\,e^{Mkt} - AMe^{Mkt}(AMke^{Mkt})}{(Ae^{Mkt}+1)^2}$

$\dfrac{dP}{dt} = \dfrac{\cancel{A^2M^2k\,e^{Mkt}} + AM^2k\,e^{Mkt} - \cancel{A^2M^2k\,e^{2Mkt}}}{(Ae^{Mkt}+1)^2} = \dfrac{AM^2k\,e^{Mkt}}{(Ae^{Mkt}+1)^2}$

$$kP(M-P) = \frac{k \cdot AMe^{Mkt}}{Ae^{Mkt}+1}\left(M - \frac{AMe^{Mkt}}{Ae^{Mkt}+1}\right)$$

$$= \frac{kAMe^{Mkt}}{Ae^{Mkt}+1}\left(\frac{\cancel{AMe^{Mkt}} + M - \cancel{AMe^{Mkt}}}{Ae^{Mkt}+1}\right)$$

$$kP(M-P) = \frac{kAM^2e^{Mkt}}{(Ae^{Mkt}+1)^2}$$

$$\therefore \quad \frac{dP}{dt} = kP(M-P) \quad \text{QED}$$

ii) $M = 860 \times 10^6$ $\rightarrow$ $P = \dfrac{A \times 860 \times 10^6 \times e^{860 \times 10^6 kt}}{Ae^{860 \times 10^6 \times kt} + 1}$

(Y. 1790) $t = 0$ $\quad P = 4 \times 10^6 = \dfrac{860 \times 10^6 A}{A+1}$

$$4(A+1) = 860A$$

$$4 = 856A$$

$$\therefore A = \frac{1}{214}$$

(Y. 1800) $t = 10$ $\quad P = 6 \times 10^6 = \dfrac{\frac{1}{214} \times 860 \times 10^6 \times e^{860 \times 10^6 \times 10k}}{\frac{1}{214}e^{860 \times 10^6 \times 10k} + 1}$

$$\frac{6}{214}e^{860 \times 10^6 \times 10k} + 6 = \frac{1}{214} \times 860 \times e^{860 \times 10^6 \times 10k}$$

$$e^{860 \times 10^6 \times 10k} \cdot \frac{856}{214} = 6$$

$$e^{860 \times 10^6 \times 10 k} = \frac{6 \times 214}{856} = \frac{3}{2}$$

$$860 \times 10^6 \times 10k = \ln\left(\frac{3}{2}\right)$$

$$k = \left(\ln \frac{3}{2}\right) \div (860 \times 10^6 \times 10)$$

$$k = 4.7419 \times 10^{-11} \text{ (5 s.f)}$$

Half of M = 430

$$4\cancel{30} \times \cancel{10^6} = \frac{\frac{1}{214} \times \overset{2}{\cancel{860}} \times \cancel{10^6} \times e^{860 \times 10^6 \times 4.7419 \times 10^{-11} t}}{\frac{1}{214} e^{860 \times 10^6 \times 4.7419 \times 10^{-11} t} + 1}$$

$$\frac{1}{214} e^{860 \times 4.7419 \times 10^{-5} t} + 1 = \frac{2}{214} e^{860 \times 10^{-5} \times 4.7419 t}$$

$$1 = \frac{1}{214} e^{860 \times 10^{-5} \times 4.7419 t}$$

$$\ln(214) = (860 \times 10^{-5} \times 4.7419) t$$

$$t = \frac{\ln(214)}{860 \times 10^{-5} \times 4.7419}$$

$$t \doteq 132$$

ie. $1790 + 132 = 1922$

Yr 1922

iii) Since $\frac{dP}{dt} = kP(M-P)$, as $P \to P$, $\frac{dP}{dt} \to 0$