Student Name _____
AM KM

KAMBALA

Mathematics Extension 1

HSC Assessment Task 3

June 24, 2008

Time Allowed: 50 minutes

INSTRUCTIONS

- This task contains three questions. Marks for each part of each question are shown.
- Answer all questions in the writing booklets provided. **Start each question in a new booklet.**
- Calculators may be used.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.
- More marks will be awarded for questions involving higher order thinking skills.

Question 1 ***(start a new booklet)*** **Marks**

(a) Use the substitution $u = 1 - 2x$ to evaluate **(3)**

$$\int_0^1 x(1-2x)^4 dx$$

(b) Using the substitution $u = e^x$, or otherwise, find **(3)**

$$\int_0^{\ln 5} \frac{1}{1+e^{-x}}\, dx$$

(c) (i) Express $\sin x - \sqrt{3}\cos x$ in the form $A\sin(x-\alpha)$, with $A > 0$ and $0 < \alpha < \frac{\pi}{2}$. **(2)**

(ii) Find the general solution to $\sin x - \sqrt{3}\cos x = \frac{2}{\sqrt{2}}$. **(2)**

Question 2 ***(start a new booklet)*** **Marks**

(a) If $\tan\theta = m$ and $\tan\phi = 3$, find the value of m if $\theta - \phi = \frac{\pi}{4}$. **(2)**

(b) Find the equation of the tangent to $y = \tan^{-1}(ax+b)$ at the point where it crosses the x-axis. **(3)**

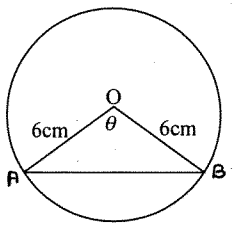
(c) Sketch the graph of $y = \frac{3}{\pi}\cos^{-1}\frac{x}{2}$ for the domain $-2 \le x \le 2$. **(2)**

(d) Find the exact value of $\int_0^3 \frac{1}{\sqrt{12-x^2}}\, dx$. **(2)**

(e) What is the condition for the inverse of a function to exist? **(1)**

Question 3 ***(start a new booklet)*** **Marks**

(a) O is the centre of a circle of radius $6cm$. **(3)**
$\angle AOB = \theta$ radians and θ is increasing at a rate of 0.2 radians per second.



Describe the rate of change of the area of ΔAOB when $\angle AOB = 120^0$.

(b) A particle, when x metres from the origin on a straight line, has velocity v ms^{-1}, given by $v^2 = 20 + 16x - 4x^2$.

(i) Show that at this point, its acceleration is $-4(x-2)$ ms^{-2}. **(1)**

(ii) State the period of the motion and find its amplitude. **(3)**

(c) A particle moves on the x-axis so that, when x metres from the origin, its acceleration is $-2x^3$ ms^{-2}.
The particle is initially at rest at $x = 4$.

(i) In what direction will the particle first move, and why? **(1)**

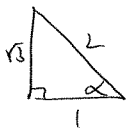
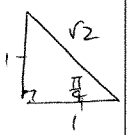
(ii) Show that its velocity, v ms^{-1}, is given by $v^2 = 256 - x^4$. **(2)**

(iii) Where will the particle next come to rest and what will be the subsequent motion? **(2)**

End of assessment task

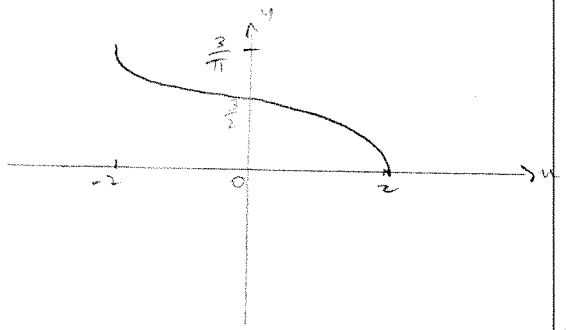
Year 12 Extension 1 Task 3 June 2008: Solutions

Qn	Solutions	Marks	Comments+Criteria
1	(a) $\int_0^1 x(1-2x)^4\,dx$ $u = 1-2x \rightarrow x = \frac{u-1}{-2}$ $du = -2\,dx$ $x=1,\ u=1-2=-1$ $x=0,\ u=1-0=1$ $I = -\frac{1}{2}\int_1^{-1}(u-1)u^4\,du$ $= \int_1^{-1} u^5 - u^4\,du$ $= \left[\frac{u^6}{6} - \frac{u^5}{5}\right]_1^{-1}$ $= \left(\frac{1}{6}+\frac{1}{5}\right) - \left(\frac{1}{6}-\frac{1}{5}\right)$ $= \frac{2}{5}$	1 1 1	
	(b) $\int_0^{\ln 5} \frac{1}{1+e^{-x}}\,dx$; $u = e^x \rightarrow u^{-1} = (e^x)^{-1} = e^{-x}$ $\frac{du}{dx} = e^x$ $du = e^x\,dx \rightarrow dx = \frac{du}{e^x} = \frac{du}{u}$ $x = \ln 5,\ u = e^{\ln 5} = 5$ $x = 0,\ u = e^0 = 1$ $I = \int_1^5 \frac{1}{1+u^{-1}} \cdot \frac{du}{u}$ $= \int_1^5 \frac{1}{u+1}\,du$ $= \left[\ln(u+1)\right]_1^5$ $= \ln_e 6 - \ln 2$		

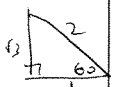
Qn	Solutions	Marks	Comments+Criteria
1	(c) (i) $\sin x - \sqrt{3}\cos x \equiv A\sin(x-\alpha)$ $A = \sqrt{a^2+b^2}$ $a=1,\ b=\sqrt{3}$. $= \sqrt{1+3}$ $= \sqrt{4}$ $= 2$ $\tan\alpha = \frac{b}{a} = \frac{\sqrt{3}}{1}$  $\alpha = \frac{\pi}{3}$ $\therefore \sin x - \sqrt{3}\cos x \equiv 2\sin\left(x-\frac{\pi}{3}\right)$		
	(ii) $\sin x - \sqrt{3}\cos x = \frac{2}{\sqrt{2}}$ (S) + (A) $2\sin\left(x-\frac{\pi}{3}\right) = \frac{2}{\sqrt{2}}$ $\sin\left(x-\frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$  basic angle, $x-\frac{\pi}{3} = \frac{\pi}{4}$. $\theta = n\pi \pm (-1)^n \alpha$ $x-\frac{\pi}{3} = n\pi \pm (-1)^n \frac{\pi}{4}$ $x = n\pi \pm (-1)^n \frac{\pi}{4} + \frac{\pi}{3}$ if n odd, $x = n\pi - \frac{\pi}{4} + \frac{\pi}{3}$ $= n\pi + \frac{\pi}{12}$ if n even or 0, $x = n\pi + \frac{\pi}{4} + \frac{\pi}{3}$ $x = n\pi + \frac{7\pi}{12}$		

Year 12 Extension 1 Task 3 June 2008: Solutions

Qn	Solutions	Marks	Comments+Criteria
2	(a) $\tan\theta = m$, $\tan\phi = 3$ $\theta - \phi = \frac{\pi}{4}$ $\tan(\theta-\phi) = \tan\frac{\pi}{4}$ $\therefore \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi} = 1$ ie $\frac{m-3}{1+3m} = 1$ $m-3 = 1+3m$ $-2m = 4$ $m = -2$		
	(b) $y = \tan^{-1}(ax+b)$ $\frac{dy}{dx} = \frac{1}{1+(ax+b)^2} \cdot a$ $= \frac{a}{1+(ax+b)^2}$ when $y=0$, $\tan^{-1}(ax+b) = 0$ $\tan 0 = ax+b$ $\frac{dy}{dx} = \frac{a}{1+(\tan 0)^2}$ $= a$ $y-0 = a(x-(ax+b))$ $= a(x-ax-b)$ $y = ax - a^2x - ab$		

Qn	Solutions	Marks	Comments+Criteria
2	(c) $y = \frac{3}{\pi}\cos^{-1}\frac{x}{2}$ for $-2 \le x \le 2$ 		
	(d) $\int_0^3 \frac{1}{\sqrt{12-x^2}}\,dx$ $\quad \sqrt{12} = 2\sqrt{3}$. $= \int_0^3 \frac{1}{\sqrt{(2\sqrt{3})^2 - x^2}}\,dx$ $= \left[\sin^{-1}\left(\frac{x}{2\sqrt{3}}\right)\right]_0^3$ $= \sin^{-1}\left(\frac{3}{2\sqrt{3}}\right) - \tan^{-1}0$ $= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \tan^{-1}0$ $= \frac{\pi}{3}$		
	(e) ~~If~~ $y = f(x)$, then $y = f^{-1}(x)$ exists iff ~~for~~ every x value there exists one and only one value for y.		

Year 12 Extension 1 Task 3 June 2008: Solutions

Qn	Solutions	Marks	Comments+Criteria
3	(a) $\frac{dr}{dt} = 0.2$ m/sec $\frac{dA}{dt} = ?$ when $\theta = 120°$ $A = \frac{1}{2}r^2\sin\theta$ $\frac{dA}{dr} = r\sin\theta$ $\frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt}$ $= r\sin\theta\cdot\frac{dr}{dt}$ $= 6\sin 120°\cdot 0.2$ $= 6\cdot\frac{\sqrt{3}}{2}\cdot 0.2$ $= 3\sqrt{3}\cdot 0.2$ $= \frac{3\sqrt{3}}{5}$ 		
	(b) $v^2 = 20 + 16x - 4x^2$ (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(10 + 8x - 2x^2\right)$ $= 8 - 4x$ $= -4(x-2)$ m/s^2		
	(ii) $\ddot{x} = -4(x-2)$ $n^2 = 4$ $n = 2$ $\therefore$ period $= \frac{2\pi}{2} = \pi$		

Qn	Solutions	Marks	Comments+Criteria
3	(c) $a = -2x^3$ m/s^2 $t=0$, $x=4$		
	(i) Initially, the particle will move left of $x=4$ (in neg. direction) since $a=-2<0$.		
	(ii) $v^2 = 256 - x^4$		$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
	$a = -2x^3$		$= \frac{d}{dx}\left(\frac{1}{2}(256-x^4)\right)$
	$v = -\frac{2x^4}{4} + C$		$= \frac{d}{dx}\left(128 - \frac{x^4}{2}\right)$
	$= -\frac{x^4}{2} + C$		$= -\frac{4x^3}{2}$
	when $t=0$, $v=0$; $x=4$.		$= -2x^3$.
	$0 = -\frac{4^4}{2} + C$		
	$= -\frac{256}{2} + C$		
	$\therefore C = 128$		
	$v = -\frac{x^4}{2} + 128 \rightarrow$		$v^2 = \left(-\frac{x^4}{2} + 128\right)^2$
			$= \frac{x^8}{2} - 128x^4 + 128^2$
	$v = \frac{1}{2}(x^4 - 256)$		
	$v^2 = \left[-\frac{1}{2}(x^4 - 256)\right]^2$		
	$= \frac{1}{4}(x^4 - 256)^2$		
	$= \frac{1}{4}(x^8 - 512x^4 + 256^2)$		
	$= \frac{1}{4}x^8 - 128x^4 + 16384)$		

Year 12 Extension 1 Task 3 June 2008: Solutions

Qn	Solutions	Marks	Comments+Criteria
3	(c) ~~c+d~~		
	(iii) $v^2 = 256 - x^4$		
	$v = \sqrt{256 - x^4}$		
	when $v=0$, $256 - x^4 = 0$		
	$x^4 = 256$		
	$x = \pm 4$		
	$\therefore$ particle at rest initially at $x=4$ and again at $x=-4$.		
	at $x=-4$, $a = -2\times(-4)^3$		
	$= 2 \times -64$		
	$= -128 < 0$		
	$\therefore$ particle is moving in neg dirn		