Total marks (36)
Attempt questions 1 – 3
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the expansion of $\left(3x^2 - \frac{1}{x}\right)^9$.

(i) How many terms are there in this expansion? **1**

(ii) Find the constant term in this expansion. **3**

(b) By equating the coefficients of x^5 on both sides of the identity

$$(1+x)^4(1+x)^4 = (1+x)^8$$

prove that $\quad {}^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4 = \dfrac{8!}{3! \times 5!}$ **4**

(c) Using the expansion of $(1+x)^n$ prove:

(i) $10^n = \binom{n}{0} + 3^2.\binom{n}{1} + 3^4.\binom{n}{2} + ... + 3^{2n}.\binom{n}{n}$ **1**

(ii) Hence, show that $1 + 3^4.\binom{n}{2} + 3^8.\binom{n}{4} + ... + 3^{2n}.\binom{n}{n} = 2^{n-1}(5^n + 4^n)$,
where n is an even integer. **3**

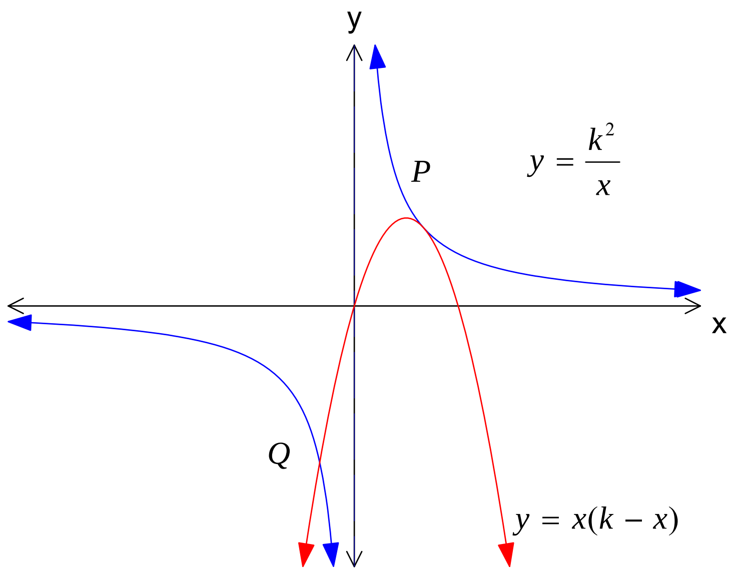
Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Find the linear factors of $6 + 5x - 2x^2 - x^3$. **2**

(ii) Find the values of x for which $6 + 5x - 2x^2 - x^3 > 0$. **2**

(b) The polynomial $P(x) = 2x^3 + ax^2 + bx + 6$ has $(x - 1)$ as a factor and leaves a remainder of -12 when divided by $(x + 2)$. Find the values of a and b. **3**

(c)

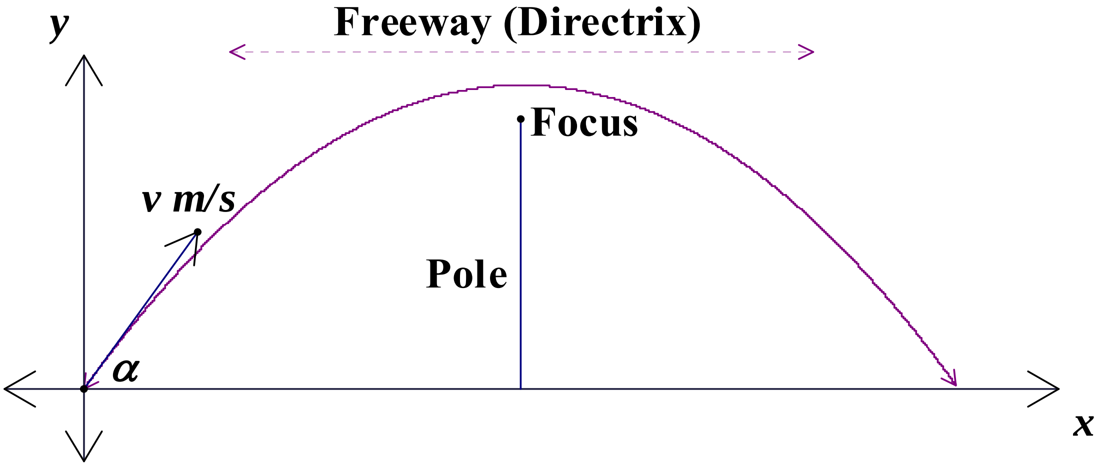


In the diagram above, the curves $y = \dfrac{k^2}{x}$ and $y = x(k - x)$, where $k > 0$, touch at the point P and intersect at the point Q.

(i) Explain why the equation $x^3 - kx^2 + k^2 = 0$ has real roots α, α, and β for some $\alpha \neq \beta$? **2**

(ii) Find the exact values of k, α and β. **3**

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**



A frog jumps with initial velocity v m/s at an angle of projection α and its path traces a parabolic arc as shown above. The frog's horizontal displacement from the origin, t seconds after jumping is given by the equation $x = vt\cos\alpha$ [Do not prove this]. The frog's vertical displacement from the origin, t seconds after jumping is given by the equation $y = vt\sin\alpha - \frac{1}{2}gt^2$ [Do not prove this].

(i) Show that the frog lands after $\dfrac{2v\sin\alpha}{g}$ seconds. **2**

(ii) Show that the frog's range is $\dfrac{v^2\sin 2\alpha}{g}$ metres. **2**

(iii) Show that the frog's maximum height is $\dfrac{v^2\sin^2\alpha}{2g}$ metres. **2**

(iv) Show that the ratio of the frog's maximum height to range is $\dfrac{\tan\alpha}{4}$. **2**

(v) Let $\alpha_{\max}$ be the angle the frog jumps at to ensure maximum range. Find the ratio of the frog's maximum height to its range in this case. **1**

(vi) Let $g = 10m/s^2$ and $v = \sqrt{85}\,m/s$.
Let α_{equal} be the angle where the frog's maximum height equals its range.
At this angle, the frog needs a 40cm gap to squeeze between a pole and a freeway.
The top of the pole is the focus of the parabola and the freeway is the directrix.
Can the frog squeeze through? **3**

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

Ext.1 Ass3 2011

Q1. (a) (i) 10 terms

(ii) $\left(3x^2 - \frac{1}{x}\right)^9 = \sum_{k=0}^{9} (3x^2)^k \left(\frac{-1}{x}\right)^{9-k} \, {}^9C_k$

$= \sum_{k=0}^{9} {}^9C_k \, 3^k \, x^{2k} (-1)^{9-k} \, x^{k-9}$

$= \sum_{k=0}^{9} {}^9C_k \, (-1)^{9-k} \, 3^k \, x^{3k-9}$

for the constant term : $3k-9=0$

$\therefore k=3$

$\therefore$ coefficient $= (-1)^6 \, {}^9C_3 \, . \, 3^3$

$= 2268$

(b) LHS $= \left({}^4C_0 + {}^4C_1 x + {}^4C_2 x^2 + {}^4C_3 x^3 + {}^4C_4 x^4\right)^2$

$\therefore$ co-eff. of $x^5 = {}^4C_1 \times {}^4C_4 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_2 + {}^4C_4 \times {}^4C_1$

$= {}^4C_1 \times {}^4C_0 + {}^4C_2 \times {}^4C_1 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4$

(since ${}^nC_r = {}^nC_{n-r}$)

RHS $= \sum_{k=0}^{8} x^k \, . \, {}^8C_k$

$\therefore$ co-eff. of $x^5 = {}^8C_5$

$= \frac{8!}{(8-5)!\,5!}$

$= \frac{8!}{3!\,5!}$

$\therefore {}^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4 = \frac{8!}{3!\,5!}$

(c) (i) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$

let $x = 9 = 3^2$

$\therefore \quad 10^n = \binom{n}{0} + \binom{n}{1}3^2 + \binom{n}{2}3^4 + \dots + \binom{n}{n}3^{2n} \quad (1)$

(ii) let $x = -9 = -3^2$

$\therefore \quad (-8)^n = \binom{n}{0} - \binom{n}{1}3^2 + \binom{n}{2}3^4 - \dots + \binom{n}{n}3^{2n} \quad (2)$

but since n is even then $(-8)^n = 8^n$

adding (1) and (2) :

$2\binom{n}{0} + 2\binom{n}{2}3^4 + \dots + 2\binom{n}{n}3^{2n} = 10^n + 8^n$

$\therefore \binom{n}{0} + \binom{n}{2}3^4 + \dots + \binom{n}{n}3^{2n} = \frac{10^n + 8^n}{2}$

$= \frac{2^n.5^n + 2^n.4^n}{2}$

$= 2^{n-1}.5^n + 2^{n-1}.4^n$

$= 2^{n-1}(5^n + 4^n)$

Solutions

Q2

(a) (i) $P(x) = 6 + 5x - 2x^2 - x^3$

$P(-1) = 6 - 5 - 2 + 1$

$= 0$

$\therefore (x+1)$ is a ~~factor~~. 1 mark for factor thm.

$$\begin{array}{r|l} & -x^2 - x + 6 \\ \hline x+1 & -x^3 - 2x^2 + 5x + 6 \\ & -x^3 - x^2 \\ \hline & \quad -x^2 + 5x \\ & \quad -x \;\; - x \\ \hline & \qquad 6x + 6 \\ & \qquad 6x + 6 \end{array}$$

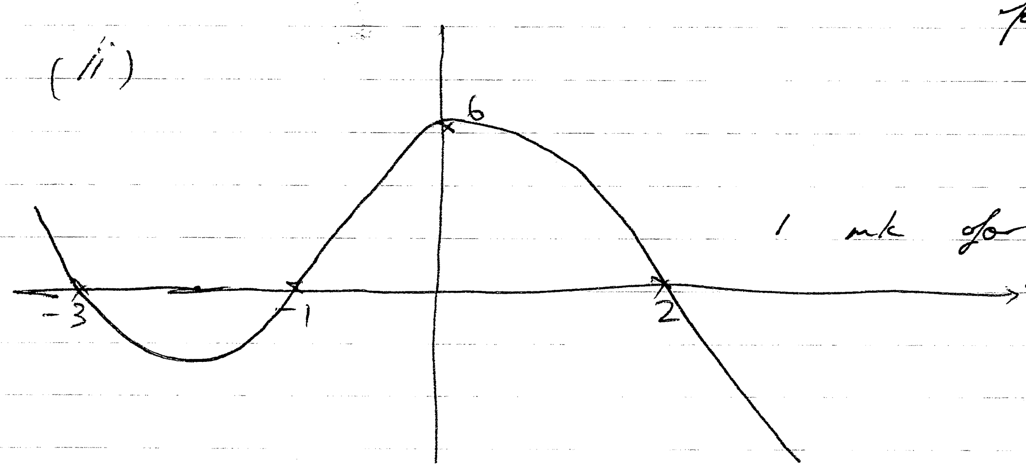
$\Rightarrow 6 - x - x^2$

$= (x+3)(2-x)$.

$\therefore P(x) = (3 + x)(2 - x)(x+1)$.

$= (x+1)(x+3)(2-x)$. 1 mk for continued factorisation

(ii)



1 mk for critical value method or graph.

$\therefore P(x) > 0$ if $x < -3$ or $-1 < x < 2$.

1 mk for values

(b) $P(x) = 2x^3 + ax^2 + bx + 6$

$P(1) = 2 + a + b + 6$

$= a + b + 8$. $\therefore a + b = -8 \quad \ldots (1)$ 1mk

$P(-2) = -16 + 4a - 2b + 6$

$= 4a - 2b - 10$

$\therefore 4a - 2b - 10 = -12$

$\therefore$ ~~2a+b~~

$4a - 2b = -2$

$2a - b = -1 \quad \ldots (2)$. 1mk.

$\therefore 3a = -9$

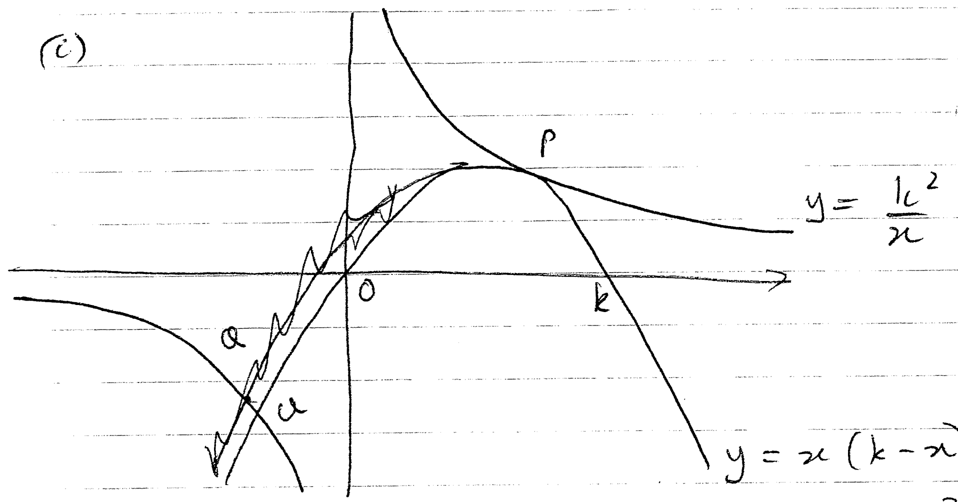
$a = \cancel{-\frac{9}{2}}\ -3.$

$b = \cancel{\frac{7}{2}}\ -5$

~~$b = -\frac{7}{2}$~~.

1mk both correct.

(c)



$y = xk - x^2$

(i) If $xk - x^2 = \frac{k^2}{x}$

then $x^2k - x^3 = k^2$,

$\therefore x^3 - \cancel{k}x^2 + k^2 = 0$.

ie. the intersection of the two curves. 1

$\therefore$ double root at ~~x=~~ P and linear root at Q

hence α, α, β are the roots. 1

(ii) $\cancel{\alpha^2 + \beta = k^2}$ $\qquad \alpha^2\beta = -k^2$. . . (1)

and $\qquad 2\alpha + \beta = k$ . . . . (2)

and $\qquad \alpha^2 + 2\alpha\beta = 0$.

~~and~~ $\qquad \alpha(\alpha + 2\beta) = 0$ . . . . (3)

if $\alpha = 0$ the β i the only solution so $\alpha \neq 0$.

$\therefore \alpha = -2\beta$. $\qquad \therefore -4\beta + \beta = k$

$\cancel{\beta = k + 4\beta}$ $\qquad -3\beta = k$

$\therefore \cancel{-3\beta = k}$ $\qquad \therefore \beta = -\frac{k}{3}$ 1

$\therefore \cancel{\beta = -\frac{k}{3}}$. $\qquad \therefore \alpha = \frac{2k}{3}$ 1

$\therefore \cancel{\alpha = -\frac{2k}{3}}$.

Now

$\alpha^2\beta = k^2$

$\therefore \frac{4k^2}{9} \cdot \left(-\frac{k}{3}\right) = k^2$

$\therefore -4k^3 = 27k^2$

$\therefore k = -\frac{27}{4}$.

$\cancel{\frac{4k^2}{9} \cdot \frac{k}{3} = k^2}$

$\therefore \cancel{4k^3 - 27k^2 = 0}$

$\therefore \cancel{k^2(4k - 27) = 0}$

$\therefore \cancel{k = \frac{27}{4}}$

$\therefore \cancel{\alpha = -\frac{27}{2}}, \cancel{\beta = \frac{25}{4}}$ 1

$\therefore \alpha = -\frac{9}{2}$.

$\beta = \frac{9}{4}$.

QUESTION THREE

(i) Frog lands when $y=0$

$\therefore \; Vt\sin\alpha - \frac{1}{2}gt^2 = 0$ ✓

$\therefore \; t(V\sin\alpha - \frac{1}{2}gt) = 0$

$\therefore \; t=0$ or $t = \frac{2V\sin\alpha}{g}$ ✓

$\therefore$ Frog lands when $t = \frac{2V\sin\alpha}{g}$

(ii) When $t = \frac{2V\sin\alpha}{g}$

$$x = Vt\cos\alpha$$
$$= V\cos\alpha\left(\frac{2V\sin\alpha}{g}\right) ✓$$
$$= \frac{V^2}{g}(2\sin\alpha\cos\alpha)$$
$$= \frac{V^2\sin 2\alpha}{g} ✓$$

(iii) maximum height occurs when $\dot{y}=0$

$$\dot{y} = V\sin\alpha - gt$$
$$= 0 \text{ when } t = \frac{V\sin\alpha}{g} ✓$$

$$y = Vt\sin\alpha - \frac{1}{2}gt^2$$
$$= \frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g}$$
$$= \frac{V^2\sin^2\alpha}{2g} ✓$$

(iv) $\frac{\text{height}}{\text{range}} = \frac{\cancel{V^2}\sin^2\alpha}{2\cancel{g}} \times \frac{\cancel{g}}{\cancel{V^2}\sin 2\alpha}$ ✓

$$= \frac{\sin^2\alpha}{4\sin\alpha\cos\alpha}$$
$$= \frac{\tan\alpha}{4} ✓$$

(v) maximum range occurs when $\sin 2\alpha = 1$

i.e. when $\alpha = \frac{\pi}{4}$

$\therefore \; \frac{\text{height}}{\text{range}} = \frac{1}{4}$ ✓

(vi) If maximum height = range then $\tan\alpha = 4$

$x = Vt\cos\alpha \quad \therefore \; t = \frac{x}{V\cos\alpha}$

$$y = Vt\sin\alpha - \frac{1}{2}gt^2$$
$$= V\sin\alpha\left(\frac{x}{V\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2$$
$$= x\tan\alpha - \frac{1}{2}\frac{gx^2}{V^2}\sec^2\alpha$$

$$\therefore \; y = 4x - \frac{5x^2}{85} \times 17$$
$$y = 4x - x^2 ✓$$

$$x^2 - 4x + 4 = -y + 4$$
$$(x-2)^2 = -(y-4)$$

$\therefore$ vertex $(2,4)$ focal length $\frac{1}{4}$

focus $(2, 3\frac{3}{4})$ directrix: $y = 4\frac{1}{4}$ ✓

$\therefore$ distance between focal length and directrix is $\frac{1}{2}$ metre > 40cm

$\therefore$ Frog can squeeze through ✓

N.B. Had to get to eqn. of parabola for first mark.

12