Total marks (41)

Attempt questions 1 – 4

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) **Marks**

a) Use the standard integrals to show that: **2**

$$\int_6^{15} \frac{dx}{\sqrt{x^2+64}} = \ln 2$$

b) Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 2x\,dx$ (Leave your answer in exact form.) **3**

c) Find $\int_1^4 2x\sqrt{1+3x^2}\,dx$ (Use the substitution $u = 1+3x^2$) **3**

d) Show that $\frac{d}{dx}\left(x^3e^{x^3}\right) = 3x^2e^{x^3} + 3x^5e^{x^3}$ and use this result to evaluate **3**

$$\int_0^1 x^5e^{x3}dx.$$

e) Let $f(x) = x + \log_e x$

i. Show that the curve $y = f(x)$ cuts the x axis between $x = 0.5$ and $x = 1$. **2**

ii. Use one application of Newtons Method with a first approximation to find a second approximation to the root of $x + \log_e x = 0$, to 3 decimal places. **2**

Question 2 (8 marks) Begin a new page. Marks

a) Prove that $\dfrac{d^2x}{dt^2} = \dfrac{d}{dx}\left(\dfrac{v^2}{2}\right)$. 2

b) A particle is moving in a straight line with $v^2 = 36 - 4x^2$.

i. Prove the particle is moving in simple harmonic motion. 2

ii. What is the amplitude and period of the motion? 2

iii. If the particle is initially at the origin write an expression for its displacement in terms of t. 2

Question 3 (10 marks) Begin a new page.

a) A particle is moving about the origin. Its displacement x centimetres from 0 at time t seconds is given by $x = 3\cos\left(2t + \dfrac{\pi}{3}\right)$.

i. Show that the particle is moving under simple harmonic motion. 2

ii. State the amplitude and period of the motion. 2

iii. Find the maximum speed of the motion and when it first occurs. 2

iv. Find the displacement of the particle when it is first at its maximum speed. 2

v. Sketch the displacement-time graph, $0 \le t \le \pi$. 2

Question 4 (8 marks) Begin a new page.

a) Fred Kicks a football from ground level at an angle of $72°$ to the horizontal with velocity $30ms^{-1}$. With the origin set where the ball was kicked an using $g = 9.8ms^{-2}$;

i. Show the equations of motion are , **2**

$$x = 30t\cos 72°$$
$$y = 30t\sin 72^{\circ} - 4.9t^2$$

ii. Calculate the time that the ball is in the air before returning to the ground, to 2 decimal places. **2**

iii. What is the maximum height the ball reaches, to 3 decimal places? **1**

iv. If the ball was kicked at an angle of 60° instead of 72°, what change would this make to the distance the ball travelled horizontally before returning to the ground, to 3 decimal places. **3**

Extension 1 Assessment 1 Solutions.

Question 1

a)

$$\int_{6}^{15} \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int_{6}^{15} \frac{dx}{\sqrt{x^2+64}} = \left[\ \ln\left(x+\sqrt{x^2+64}\right)\ \right]_{6}^{15}$$

$$= \left[\ \ln\left(15+\sqrt{15^2+64}\right) - \ln\left(6+\sqrt{6^2+64}\right)\right]$$

$$= \ln 32 - \ln 16$$

$$= \ln\frac{32}{16}$$

$$= \ln 2$$

b) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 2x dx$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}\cos 4x + \frac{1}{2} dx$$

$$= \left[\frac{1}{8}\sin 4x + \frac{1}{2}x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \left[\left(\frac{1}{8}\sin 2\pi + \frac{\pi}{4}\right) - \left(\frac{1}{8}\sin\frac{4\pi}{3} + \frac{\pi}{6}\right)\right]$$

$$= \left[\frac{\pi}{4} - \left(\frac{1}{8}\times\left(-\sin\frac{\pi}{3}\right) + \frac{\pi}{6}\right)\right]$$

$$=\left[\frac{\pi}{4}-\left(-\frac{\sqrt{3}}{16}+\frac{\pi}{6}\right)\right]$$

$$=\frac{12\pi-8\pi+3\sqrt{3}}{48}$$

$$=\frac{4\pi-3\sqrt{3}}{48}$$

c) $\int_1^4 2x\sqrt{1+3x^2}\,dx$ $\qquad u=1+3x^2$

$$=\int_4^{49}\sqrt{u}.\frac{1}{3}du \qquad du=6xdx$$

$$=\frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_4^{49} \qquad x=1,u=4$$

$$=\frac{2}{9}\left[49^{\frac{3}{2}}-4^{\frac{3}{2}}\right]$$

$$=\frac{2}{9}[343-8]$$

$$=\frac{670}{9}$$

d) $\frac{d\left(x^2e^{x^2}\right)}{dx}=2x.e^{x^2}+x^2.2xe^{x^2}$

$$=2xe^{x^2}+x^3e^{x^2}$$

$$\int_0^1 x^3e^{x^2}\,dx=\frac{1}{2}\int_0^1\frac{d}{dx}\left(x^2e^{x^2}\right)-2xe^{x^2}\,dx$$

$$=\frac{1}{2}\left[\ \left[x^2e^{x^2}\right]_0^1-\int_0^1 2xe^{x^2}\,dx\ \right]$$

$$=\frac{1}{2}\left[\ [e-0]_0^1-\left[e^{x^2}\right]_0^1\ \right]$$

$$=\frac{1}{2}\left[e-[e-1]\ \right]$$

$$=\frac{1}{2}$$

e). i. $f(x) = x + \log_e x$

$f(0.5) = 0.5 + \log_e 0.5$

$= -0.19314718$ <0

$f(1) = 1 + \log_e 1$

$= 1$ >0

Therefore there is a root between $x = 0.5$ and $x = 1$.

ii. $$x = 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$x = 0.5 - \frac{-0.19314718}{1+2}$$

$$x = 0.564382393$$
$$x = 0.564(3dp)$$

Therefore the second approximation for the root is 0.564

Question 2

a) Prove $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$

$$\frac{d^2x}{dt^2} = \frac{dv}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{dv}{dx} \bullet \frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{dv}{dx} \bullet V$$ now $V = \frac{d}{dv}\left(\frac{1}{2}v^2\right)$

$$\frac{d^2x}{dt^2} = \frac{dv}{dx} \bullet \frac{d}{dv}\left(\frac{1}{2}v^2\right)$$

$$\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

b) i. $v^2 = 36 - 4x^2$

$\frac{1}{2}v^2 = 18 - 2x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2 \right) = \frac{d}{dx}\left(18 - 2x^2 \right)$

$\ddot{x} = -4x$

$\therefore \ddot{x} = -n^2 x$ where n=2

Therefore it moves in SHM

Ii $v^2 = 36 - 4x^2$

$v^2 = -4\left(x^2 - 9 \right)$

$v^2 = -n^2\left(x^2 - a^2 \right)$

$\therefore n = 2$ and $a = \pm 3$

Therefore the amplitude is 3 and the period is $- \frac{2\pi}{2} = \pi$

Iii $t = 0$ and $x = 0$

$v^2 = 36 - 4x^2$

$v = \sqrt{36 - 4x^2}$

$\frac{dx}{dt} = \sqrt{36 - 4x^2}$

$\frac{dt}{dx} = \frac{1}{\sqrt{36 - 4x^2}}$

$\int dt = \int \frac{1}{2\sqrt{9 - x^2}} dx$

$\therefore t = \frac{1}{2}\sin^{-1}\frac{x}{3} + C$

As $t = 0$ and $x = o$ then $C = 0$

$\therefore 2t = \sin^{-1}\frac{x}{3}$

OR consider this solution

From ii. N=2 and a=3

a possible solution is $x = 3\sin 2t$

CHECK $\dot{x} = 6\cos 2t$

$\ddot{x} = -12\sin 2t$

$\ddot{x} = -4\left(3\sin 2t\right)$

$\ddot{x} = -4x$

which satisfies the differential eqn.

$\therefore x = 3\sin 2t$ is the solution

$\sin 2t = \frac{x}{3}$

$x = 3\sin 2t$

Question 3

a) i. $x = 3\cos\left(2t + \frac{\pi}{3}\right)$

$\dot{x} = -6\sin\left(2t + \frac{\pi}{3}\right)$

$\ddot{x} = -12\cos\left(2t + \frac{\pi}{3}\right)$

$\ddot{x} = -4\left(3\cos\left(2t + \frac{\pi}{3}\right)\right)$

$\ddot{x} = -4x$ where $x = 3\cos\left(2t + \frac{\pi}{3}\right)$

$\ddot{x} = -n^2 x$

where n=2

Therefore it is moving in SHM.

Ii Amplitude= 3

Period= $\frac{2\pi}{2} = \pi$

iii. max speed = Ivelocity I

Max value of $\sin\left(2t + \frac{\pi}{3}\right)$ is ± 1

Max speed = I -6 x 1I =6m/s

$6 = \left| -6\sin\left(2t + \frac{\pi}{3}\right) \right|$

$1 = \sin\left(2t + \frac{\pi}{3}\right)$

$\therefore \left(2t + \frac{\pi}{3}\right) = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots\ldots$

$t = \frac{\pi}{12}, \frac{13\pi}{12}, \ldots$

Therefore the first time it is at max speed is $\frac{\pi}{12} s.$

iv. $x = 3\cos\left(2t + \frac{\pi}{3}\right)$,

$x = 3\cos\left(2.\frac{\pi}{12} + \frac{\pi}{3}\right)$

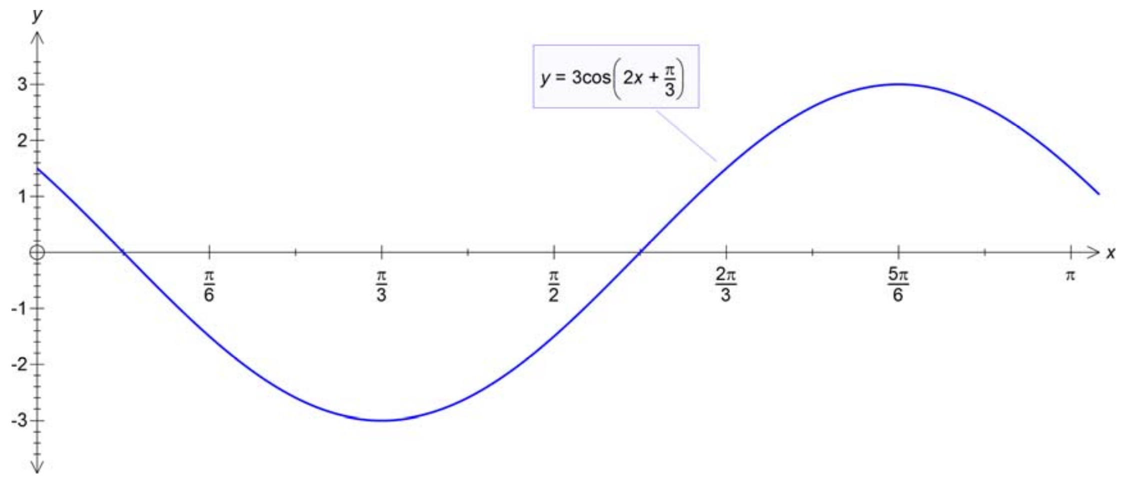
$x = 3\cos\left(\frac{\pi}{2}\right)$

$x = 0m.$

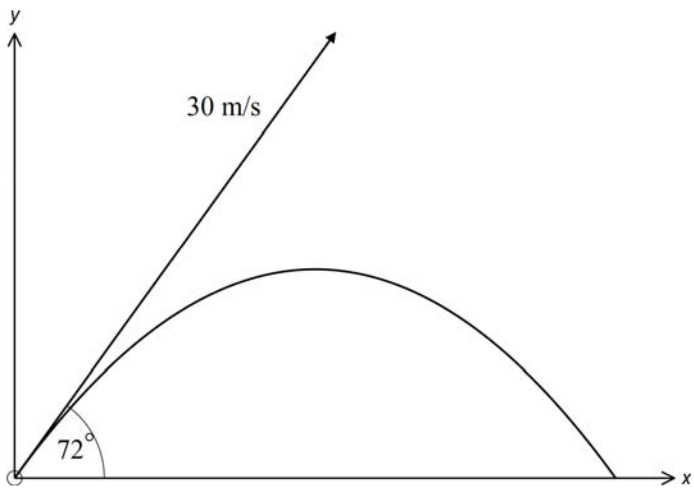
v.

Sketch

$x = 3\cos\left(2t + \frac{\pi}{3}\right)$



Question4



Initial velocity

$$\cos 72^\circ = \frac{\dot{x}}{30}$$

$$\dot{x} = 30\cos 72^\circ$$

$$\sin 72^\circ = \frac{\dot{y}}{30}$$

$$\dot{y} = 30\sin 72^\circ$$

Horizontal Motion

$$\ddot{x} = 0$$

$$\dot{x} = C_1 \qquad t = 0,\quad \dot{x} = 30\cos 72^\circ \ \therefore \therefore\ C_1 = 30\cos 72^\circ$$

$$\therefore \dot{x} = 30\cos 72^\circ$$

$$\therefore x = 30\cos 72^\circ t + C_2 \qquad t = 0, x = 0, \therefore c_2 = 0$$

$$\therefore x = 30\cos 72^\circ t$$

Vertical Motion

$\ddot{y} = -9.8$

$\dot{y} = -9.8t + C_3$ $\quad t = 0,\ \dot{y} = 30\sin 72^\circ,\ C_3 = 30\sin 72^\circ$

$\dot{y} = -9.8t + 30\sin 72^\circ$

$y = -4.9t^2 + 30\sin 72^\circ t + C_4$ $\quad t = 0, y = 0, \therefore c_4 = 0$

$\therefore y = -4.9t^2 + 30\sin 72^\circ t$

b) $y = 0,\ t = ?$

$\therefore -4.9t^2 + 30\sin 72^\circ t = 0$

$\therefore t\left(-4.9t + 30\sin 72^\circ \right) = 0$

$\therefore t = 0$ or $\therefore -4.9t + 30\sin 72^\circ = 0$

$\therefore t = \frac{30}{49}\sin 72^\circ$

$t = 5.82279499$

$t = 5.82s$

Therefore the ball returns to the ground after 5.82 seconds.

c) Max height

Max height occurs at $t = 5.82279499 \div 2s$

$t = 2.91139749s$

$\therefore y = -4.9(2.91139749)^2 + 30\sin 72^\circ (2.91139749)$

$y = 41.5335344$

$y = 42m(n.m)$

d) horizontal distance difference.

$\therefore x = 30\cos 72^\circ t \quad t = 5.82279499$

$\therefore x = 30\cos 60^\circ \times 5.82279499$

$x = 87.3419248$

$\therefore x = 30\cos 60^\circ t$

$\therefore x = 30\cos 72^\circ \times 5.82279499$

$x = 50.04247004$

Difference=37.29945481

= 37m (n.m)

Therefore the difference in length is 37m