NORTH SYDNEY GIRLS HIGH SCHOOL

YEAR 12 – TERM 2 ASSESSMENT

2007

MATHEMATICS EXTENSION 1

TIME ALLOWED: 60 minutes
Plus 2 minutes reading time

INSTRUCTIONS:

- Start each question on a new page
- Hand each question in separately, including a sheet for non-attempts
- Show all necessary working

This task is worth 25% of the HSC Assessment Mark

QUESTION 1. (7 marks) Start a new page **Marks**

a) Find the following indefinite integrals:

(i) $\int \frac{dx}{x^2}$ **1**

(ii) $\int (2x+5)^4 dx$ **2**

b) Find the equation of the curve given $\frac{d^2y}{dx^2} = 2x - 1$. **4**

and when $x = 3$, $\frac{dy}{dx} = 8$, and the point $(0, -1)$ lies on the curve.

QUESTION 2. (8 marks) Start a new page

a) Convert $\frac{2\pi}{3}$ radians to degrees. **1**

b) Find the area of a sector of a circle of radius 8 cm if the length of the arc is 2 cm. **3**

c) Use Simpson's rule with 5 function values to approximate $\int_1^5 \frac{dx}{x+1}$. **4**

QUESTION 3. (8 marks) Start a new page

a) Differentiate with respect to x:

(i) e^{3x} **1**

(ii) $\ln 3x$ **1**

(iii) 3^x **2**

b) Differentiate $e^{2x}\cos x$ **2**

c) Evaluate $\lim_{x \to 0} \frac{\sin 4x - \sin x}{2x}$. **2**

QUESTION 4. (7 marks) Start a new page

a) (i) State the period and amplitude of $y = 3\sin\frac{x}{2}$. **2**

(ii) Sketch the curve $y = 3\sin\frac{x}{2}$ for $-\pi \le x \le \pi$. **2**

b) Solve $\tan 2x = -\sqrt{3}$ for $0 \le x \le 2\pi$. **3**

continued on next page

QUESTION 5. (7 marks) Start a new page **Marks**

a) Find $\int_0^{\frac{\pi}{8}} \cos^2 2x \, dx$. **3**

b) A person invests \$2000 at the beginning of each year in a superannuation fund. Assuming that interest is paid at the rate of 9% pa on this money,

(i) show that at the end of n years the investment will have a value of $\frac{218\,000(1 \cdot 09^n - 1)}{9}$ dollars. **2**

(ii) find the number of <u>full</u> years required for the accumulated value to reach \$100 000. **2**

QUESTION 6. (8 marks) Start a new page

a) Using the substitution $u = 2 + x^2$, find $\int 3x(2 + x^2)^3 \, dx$. **3**

b) (i) Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$. **2**

(ii) Given that $y = (\sec x + \tan x)^m$ and m is a positive integer show that $\frac{dy}{dx} = my \sec x$. **3**

End of paper

Year 12 Term 2 2007 Extension 1

Question 1

a i) $\int x^{-2}\,dx = -\frac{1}{x} + C$ (1)

ii) $\int (2x+5)^4\,dx = \frac{(2x+5)^5}{10} + c$ (2)

b) $\frac{d^2y}{dx^2} = 2x - 1$

$\frac{dy}{dx} = x^2 - x + C$

$8 = 3^2 - 3 + C$

$C = 2$

$\frac{dy}{dx} = x^2 - x + 2$

$y = \frac{x^3}{3} - \frac{x^2}{2} + 2x + D$

$-1 = 0 + D$

$y = \frac{x^3}{3} - \frac{x^2}{2} + 2x - 1$ (4)

Question 2

a) $\frac{2\pi}{3} = \frac{2}{3} \times 180^\circ$

$= 120^\circ$ (1)

b) $2 = 8\theta \rightarrow \theta = 1/4$

$\text{area} = \frac{1}{2} \times 8^2 \times \frac{1}{4}$

$= 8\ \text{cm}^2$ (3)

c) $\int_1^5 \frac{dx}{x+1} = \int_1^3 \frac{dx}{x+1} + \int_3^5 \frac{dx}{x+1}$

x	1	2	3	4	5
$f(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$

$\doteqdot \frac{3-1}{6}\left(\frac{1}{2} + 4 \times \frac{1}{3} + \frac{1}{4}\right)$

$+ \frac{5-3}{6}\left(\frac{1}{4} + 4 \times \frac{1}{5} + \frac{1}{6}\right)$

$= 1.1$ (4)

Question 3

a i) $\frac{d}{dx}(e^{3x}) = 3e^{3x}$ (1)

ii) $\frac{d}{dx}(3^x) = \ln 3 \times 3^x$ (1)

iii) $\frac{d}{dx}\left(\ln(2x-1) - \ln(3x+1)\right)$

$= \frac{2}{2x-1} - \frac{3}{3x+1}$ (2)

b) $\frac{d}{dx}(e^{2x}\cos x) = 2e^{2x}\cos x - e^{2x}\sin x$ (2)

c) $\lim_{x \to 0}\left(\frac{\sin 4x}{2x} - \frac{\sin x}{2x}\right)$

$= \lim_{x \to 0}\left(\frac{\sin 4x}{4x} \times 2 - \frac{\sin x}{x} \times \frac{1}{2}\right)$

$= 2 - \frac{1}{2}$

$= \frac{3}{2}$ (2)

Question 4

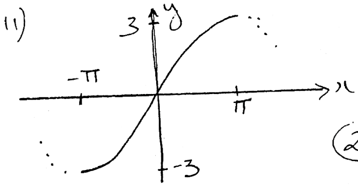
a i) $y = 3\sin\frac{x}{2}$

$\text{period} = \frac{2\pi}{\left(\frac{1}{2}\right)}$

$= 4\pi$ (2)

amplitude $= 3$

ii)



(2)

b) $\tan 2x = -\sqrt{3}$ $\quad 0 \le x \le 2\pi$

$0 \le 2x \le 4\pi$

$2x = \pi - \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}$

$3\pi - \frac{\pi}{3},\ 4\pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}, \frac{11\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$ (3)

Question 5

a) $\int_0^{\pi/8} \cos^2 2x\,dx$

$= \frac{1}{2}\int_0^{\pi/8} (\cos 4x + 1)\,dx$

$= \frac{1}{2}\left[\frac{1}{4}\sin 4x + x\right]_0^{\pi/8}$

$= \frac{1}{2}\left(\frac{1}{4}\sin\frac{\pi}{2} + \frac{\pi}{8} - \left(\frac{1}{4}\sin 0 + 0\right)\right)$

$= \frac{1}{2}\left(\frac{1}{4} \times 1 + \frac{\pi}{8}\right) = \frac{\pi + 2}{16}$ (3)

Question 5 cont

b) 1st \$2000 invested for n yrs $2000(1.09)^n$

2nd ——— $n-1$ — $2000(1.09)^{n-1}$

⋮

nth ——— 1 — $2000(1.09)$

Total $= 2000\underbrace{(1.09 + 1.09^2 + \cdots + 1.09^n)}_{GP}$

∴ after n years total value is

$$\frac{2000 \times 1.09(1.09^n - 1)}{1.09 - 1}$$

$$= \frac{2000(1.09)(1.09^n - 1)}{0.09}$$

$$= \frac{218\,000(1.09^n - 1)}{9}$$ ②

ii) $\frac{218000}{9}(1.09^n - 1) = 100\,000$

$1.09^n - 1 = 100\,000 \times \frac{9}{218\,000}$

$1.09^n = \frac{900}{218} + 1$

$= \frac{1118}{218}$

$n \log 1.09 = \log\left(\frac{1118}{218}\right)$

$n = \frac{\log\left(\frac{1118}{218}\right)}{\log(1.09)}$

$\doteqdot 18.970\ldots$ ②

∴ 19 full years are required

Question 6

a) $\int 3x(2+x^2)^3\,dx$

$u = 2 + x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\therefore x\,dx = \frac{1}{2}du$

$\int 3 \times u^3 \times \frac{1}{2}\,du$

$= \frac{3}{2}\frac{u^4}{4} + C$

$= \frac{3}{8}(2+x^2)^4 + C$ ③

bi) $\frac{d}{dx}(\sec x)$

$= \frac{d}{dx}(\cos x)^{-1}$

$= -\frac{1}{\cos^2 x} \times (-\sin x)$

$= \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$

$= \tan x \sec x$ ②

ii) $y = (\sec x + \tan x)^m$

$\frac{dy}{dx} = m(\sec x + \tan x)^{m-1}(\sec x \tan x + \sec^2 x)$

$= m\sec x(\sec x + \tan x)^{m-1}(\sec x + \tan x)$

$= m\sec x(\sec x + \tan x)^m$

$= m\sec x \times y$

$= my\sec x$ ③

Question 5

(a) $\cos 2A = 2\cos^2 A - 1 \Rightarrow 2\cos^2 A = 1 + \cos 2A$

$$\int_0^{\frac{\pi}{8}} \cos^2 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{8}} 2\cos^2 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{8}} (1+\cos 4x)\,dx$$

$$= \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{8}}$$

$$= \frac{1}{2}\left(\frac{\pi}{8} + \frac{1}{4}\sin\frac{\pi}{2}\right) - 0$$

$$= \frac{\pi}{16} + \frac{1}{8}$$

(b) (i) The first \$2000 will accumulate to $2000(1\cdot 09)^n$.

The second \$2000 will accumulate to $2000(1\cdot 09)^{n-1}$ and so on until the last \$2000 will accumulate to $2000(1\cdot 09)$

The investment will have an accumulated value of \$*A*, given by:

$$A = 2000(1\cdot 09)^n + 2000(1\cdot 09)^{n-1} + ... + 2000(1\cdot 09)$$

$$A = 2000(1\cdot 09)^n + 2000(1\cdot 09)^{n-1} + ... + 2000(1\cdot 09)$$

$$= 2000\left[\underbrace{1\cdot 09 + ... + (1\cdot 09)^{n-1} + (1\cdot 09)^n}_{a=1\cdot 09,\, r=1\cdot 09}\right]$$

$$= 2000 \times 1\cdot 09 \times \left(\frac{1\cdot 09^n - 1}{0\cdot 09}\right)$$

$$= \frac{218\,000(1\cdot 09^n - 1)}{9}$$

Alternatively

Let A_n be the amount accumulated after *n* years.

$A_1 = 2000 \times 1\cdot 09$

$A_2 = (A_1 + 2000)1\cdot 09 = 2000(1\cdot 09)^2 + 2000(1\cdot 09)$

$A_3 = (A_2 + 2000)1\cdot 09 = 2000\left[1\cdot 09 + 1\cdot 09^2 + 1\cdot 09^3\right]$

$\therefore A_n = (A_{n-1} + 2000)1\cdot 09 = 2000\left[1\cdot 09 + 1\cdot 09^2 + 1\cdot 09^3 + ... + 1\cdot 09^n\right]$

$$A_n = 2000\left[\underbrace{1\cdot 09 + ... + (1\cdot 09)^{n-1} + (1\cdot 09)^n}_{a=1\cdot 09,\, r=1\cdot 09}\right]$$

$$= 2000 \times 1\cdot 09 \times \left(\frac{1\cdot 09^n - 1}{0\cdot 09}\right)$$

$$= \frac{218\,000(1\cdot 09^n - 1)}{9}$$

(ii) $A = \dfrac{218\,000\left(1\cdot 09^n - 1\right)}{9} = 100\,000$

$\therefore 1\cdot 09^n - 1 = \dfrac{100\,000 \times 9}{218\,000} = \dfrac{450}{109}$

$\therefore 1\cdot 09^n = \dfrac{559}{109}$

$\therefore \ln\left(1\cdot 09^n\right) = \ln\left(\dfrac{559}{109}\right)$

$\therefore n \ln 1\cdot 09 = \ln\left(\dfrac{559}{109}\right)$

$\therefore n = \ln\left(\dfrac{559}{109}\right) \div \ln 1\cdot 09$

$\therefore n \approx 18\cdot 970$

So 19 years will be needed for the accumulated value to reach $100 000