NORTH SYDNEY
GIRLS HIGH SCHOOL

2008

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK #3

Mathematics Extension 1

Student Number:______________________________ **Teacher:** _______________

Student Name:________________________________

General Instructions

- Reading time – 2 minutes.
- Working time – 65 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for untidy or badly arranged work.
- Start each **NEW** question on a separate answer sheet.

Total Marks 54 Marks

- Attempt Questions 1 - 6
- All questions are of equal value.

	Q 1		Q 2		Q 3		Q4	Q5		Q6		Total
	a, b, c	d	a, c	b	a	b, c		a, c	b	a, b	c	
PE 3												/13
HE 2												/7
HE 7												/28
HE 6												/6
												/54

Total marks – 54
Attempt Questions 1 - 6
All questions are of equal value

Start each question on a SEPARATE answer sheet.

Question 1 (9 marks) **Marks**

(a) Differentiate $x^2 \ln(2x+1)$ **2**

(b) Find a primitive for $\sqrt{x} - e^{-x}$ **2**

(c) Evaluate $\int_1^3 \frac{1}{2x}\, dx$, leaving your answer as an exact value. **2**

(d) Use the substitution $u = 2x - 1$ to find $\int x(2x-1)^5\, dx$ **3**

Question 2 (9 Marks) Start a NEW answer sheet. **Marks**

(a) The letters of the word ASSININE are arranged in a row **1**
How many distinct arrangements of all of the letters are there?

(b) (i) Prove, using mathematical induction, that for all integers $n,\ n \geq 1$

$$\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$ **3**

(ii) Hence, or otherwise, evaluate $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{k(k+1)(k+2)}$ **1**

(c) There are 4 married couples at a dinner party.

The 8 people are to be arranged around a table for dinner.

(i) What is the probability that Mr Rekrap will not be seated next to Mrs Elbmal? **2**

(ii) If the seats were numbered from 1 through to 8, in how many ways can the men and women be seated alternately? **2**

Question 3 (9 Marks) Start a NEW answer sheet. **Marks**

(a) Using the substitution $u^2 = 1 - x$, evaluate $\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}\, dx$ **3**

(b) Lucy visits the boutique *La Belle Pour Elle*. **2**

She likes 3 kinds of tops, 4 different dresses, 3 pairs of shoes and 5 different skirts. All the items cost the same amount.

Lucy can only afford to buy three things.

What is the probability that she buys 2 pairs of shoes and a dress at this shop?

(c) Gilly, in retirement, is playing with the letters of the word CRICKET.

(i) By considering cases, or otherwise, how many 5-letter words can he form? **2**

(ii) What is the probability that the 5-letter word he forms will have no vowels? **2**

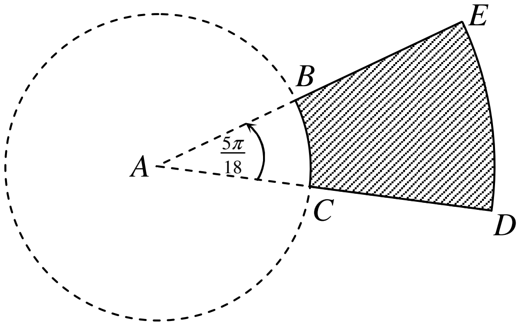
Question 4 (9 Marks) Start a NEW answer sheet. **Marks**

(a) Evaluate $\lim_{x \to 0} \dfrac{\sin^2 4x}{\tan^2 3x}$ **1**

(b) In the shape below, A is the centre of two concentric circles, one which passes through B and C and another which passes through D and E.

$\angle BAC = \dfrac{5\pi}{18}$, $AB = 1 \cdot 3$ m and $CD = 40$ m

NOT TO SCALE



(i) Find the length of arc DE. **1**

(ii) Find the shaded area, $BEDC$, bounded by the arcs BC and DE and the straight edges BE and CD. **2**

(c) (i) Sketch $y = 1 - \cos x$ for $0 \leq x \leq 4\pi$ **3**

(ii) Hence, or otherwise, find the number of values of x which satisfy the equation **2**

$$3\pi(1 - \cos x) = 2x$$

Question 5 (9 Marks) Start a NEW answer sheet. **Marks**

(a) (i) Show that $\dfrac{1}{3x-1}-\dfrac{1}{3x+1}=\dfrac{2}{9x^2-1}$ **1**

(ii) Hence, or otherwise, find $\displaystyle\int \frac{dx}{9x^2-1}$. **2**

(b) Jha Hyun's tennis ball box contains ten tennis balls of which four have never been used. **2**

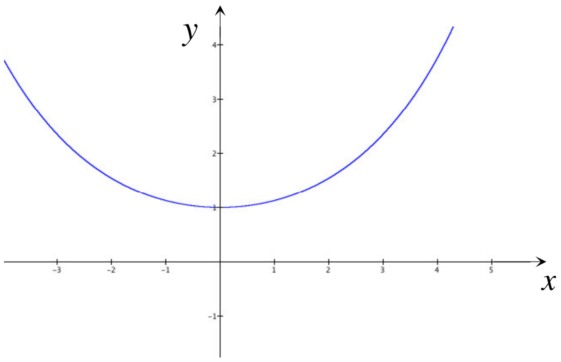
For the first game two balls are selected at random and, after play, are returned to the box.

For the second game two balls are also selected at random from the box.

Find the probability that neither ball selected for the first game has been used before, but both balls selected for the second game have been used before the second game.

(c) (i) Expand and simplify $\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)^2$ **1**

(ii) The graph of $y=\dfrac{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}{2}$ is drawn below. **3**



The area bounded by the curve and the lines $x=-3$ and $x=3$ is rotated about the x-axis.
Calculate the volume of the solid of revolution, leaving your answer as an exact value.

Question 6 (9 Marks) Start a NEW answer sheet. **Marks**

(a) A function $f(x)$ is given, for $x > 0$

$$f(x) = 2\ln x - \frac{x^2 - 1}{x}$$

(i) Show that $f(1) = 0$ **1**

(ii) Show that $f'(x) = -\left(\frac{x-1}{x}\right)^2$ **2**

(iii) Hence, or otherwise, show that the only zero of $f(x)$ is at $x = 1$. **1**

(b) Find a primitive of $\dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$ **2**

(c) (i) Given that $(k+1)! > 3^{k-1}$, for some positive integer k, prove that **1**

$$(k+2)! > 3^k$$

(ii) Hence prove that $(n+1)! > 3^{n-1}$ for all positive integers n. **2**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

S

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ASSESSMENT TASK #3

Mathematics Extension 1

Sample Solutions

Question 1

HE7

(a) $$\frac{d}{dx}\left[x^2\ln(2x+1)\right] = x^2 \times \frac{2}{2x+1} + 2x \times \ln(2x+1)$$
$$= \frac{2x^2}{2x+1} + 2x\ln(2x+1)$$ **2**

(b) $$\int \left(x^{\frac{1}{2}} - e^{-x}\right) dx = \frac{2}{3}x^{\frac{3}{2}} + e^{-x} + c$$ **2**

(c) $$\int_1^3 \frac{1}{2x}\,dx = \frac{1}{2}\int_1^3 \frac{2}{2x}\,dx$$
$$= \frac{1}{2}\left[\ln 2x\right]_1^3$$
$$= \frac{1}{2}(\ln 6 - \ln 2)$$
$$= \frac{1}{2}\ln\left(\frac{6}{2}\right)$$
$$= \frac{1}{2}\ln 3$$

OR

$$\int_1^3 \frac{1}{2x}\,dx = \frac{1}{2}\int_1^3 \frac{1}{x}\,dx$$
$$= \frac{1}{2}\left[\ln x\right]_1^3$$
$$= \frac{1}{2}(\ln 3 - \ln 1)$$
$$= \frac{1}{2}\ln 3$$ **2**

HE6

(d) $$u = 2x - 1$$
$$\therefore \frac{du}{dx} = 2$$
$$\therefore du = 2dx$$

$$u = 2x - 1$$
$$2x = u + 1$$

$$\int x(2x-1)^5\,dx = \frac{1}{4}\int 2x(2x-1)^5 \times 2dx$$
$$= \frac{1}{4}\int (u+1)u^5\,du$$
$$= \frac{1}{4}\int \left(u^6 + u^5\right) du$$
$$= \frac{1}{4}\left(\frac{u^7}{7} + \frac{u^6}{6}\right) + c$$
$$= \frac{(2x-1)^7}{28} + \frac{(2x-1)^6}{24} + c$$ **3**

Question 2 **Marks**

PE3

With ASSININE there are 8 letters, 2 S, 2 I and 2N.

So there are $\frac{8!}{2!2!2!} = 5040$ possible arrangements of the letters **1**

HE2

(b) (i) Test $n = 1$

LHS = $\frac{1}{1\times2\times3} = \frac{1}{6}$

RHS = $\frac{1}{4} - \frac{1}{2(2)(3)} = \frac{1}{6}$ **3**

So the formula is true for $n = 1$

If the formula is true for some integer $n = k$ i.e.

$$\frac{1}{1\times2\times3} + \ldots + \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)} \qquad -(*)$$

Then we prove it true for $n = k+1$ i.e.

$$\frac{1}{1\times2\times3} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} = \frac{1}{4} - \frac{1}{2(k+2)(k+3)}$$

$$\begin{aligned}
\text{LHS} &= \frac{1}{1\times2\times3} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \\
&= \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \qquad \left[\text{From } (*)\right] \\
&= \frac{1}{4} - \frac{(k+3)}{2(k+1)(k+2)(k+3)} + \frac{2}{2(k+1)(k+2)(k+3)} \\
&= \frac{1}{4} - \frac{k+3-2}{2(k+1)(k+2)(k+3)} \\
&= \frac{1}{4} - \frac{1}{2(k+2)(k+3)} \\
&= \text{RHS}
\end{aligned}$$

So if the formula is true for some integer $n = k$ it is also true for $n = k+1$

So by the principle of mathematical induction it is true for all integers n, $n \geq 1$

(ii) $$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \lim_{n\to\infty}\left(\frac{1}{4} - \frac{1}{2(n+1)(n+2)}\right) = \frac{1}{4}$$ **1**

(c) (i) With 8 people to sit around the table, sit Mr Rekrap first and then the rest can be seated in 7! ways. **2**

If we seat Rekrap next to Elbmal first, then the remainder can be seated in 6! ways. With them (i.e. Rekrap and Elbmal) possibly switching seats, the total number of ways is $2\times 6!$.

So the probability of them <u>not</u> sitting next to each other is $\dfrac{7!-2\times 6!}{7!}=\dfrac{5}{7}$.

ALTERNATIVELY:

If we seat Elbmal first, then there are 6 people who can be seated to the right of her and then 5 people who can be seated to the left of her.
This is as a result of Rekrap not sitting next to Elbmal.
The remaining people (including Rekrap) can be seated in 5! ways.

PE3

So there are $6\times 5\times 5!=3600$ ways for them to be seated.

So the probability is $\dfrac{3600}{7!}=\dfrac{5}{7}$.

(ii) If the seats are numbered, then sitting Elbmal has the choice of sitting in odd or even seats, then a choice of 4 seats (1, 3, 5, 7 **OR** 2, 4, 6, 8).
The remaining women have 3! ways of being seated. **2**
The men then have 4! ways of sitting in the remaining seats
So there are $2!4!4!=1152$ ways of doing this.

NB In (ii), as the seats are numbered, they are no longer equivalent and so even though there may be occasions where the same people are to the left and right of each other, they are in “different seats.

Question 3 **Marks**

HE6

(a) $u^2 = 1 - x \Rightarrow x = 1 - u^2$

$x = 0 \Rightarrow u = 1$

$x = \frac{3}{4} \Rightarrow u = \frac{1}{2}$

$\frac{dx}{du} = -2u$ **3**

$\therefore dx = -2udu$

$$\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}\,dx = \int_1^{\frac{1}{2}} \frac{1-u^2}{u}(-2udu)$$

$$= -2\int_1^{\frac{1}{2}} (1-u^2)\,du$$

$$= -2\left[u - \frac{u^3}{3}\right]_1^{\frac{1}{2}}$$

$$= -2\left[\left(\frac{1}{2} - \frac{1}{24}\right) - \left(1 - \frac{1}{3}\right)\right]$$

$$= \frac{5}{12}$$

PE3

(b) Lucy could choose $\binom{15}{3}$ things.

There are $\binom{3}{2}$ ways to choose the 2 pair of shoes and $\binom{4}{1}$ ways to pick the dress. So the probability of this is $\frac{\binom{4}{1}\times\binom{3}{2}}{\binom{15}{3}} = \frac{12}{455}$ **2**

(c) (i) There are 2 cases i.e. repetition or not. **2**

Case 1: (Repetition) CC R I K E T

So 3 letters need to be chosen from RIKET. This can be done in $\binom{5}{3} = 10$ ways. Then the letters can be arranged in $\frac{5!}{2!} = 60$ ways.

So the number of ways for this case is $10 \times 60 = 600$ ways.

Case 2: (No Repetition) C R I K E T

So five letters need to be arranged from the 6 letters. This can be done in ${}^6P_5 = 720$ ways.

So there are $600 + 720 = 1320$ ways that Gilly can make 5-letter words.

(ii) No vowels ie C C R K T can be arranged in $\frac{5!}{2!} = 60$ ways.

So the probability of this is $\frac{60}{1320} = \frac{1}{22}$ **2**

Question 4 **Marks**

(a) $\displaystyle\lim_{x\to 0}\frac{\sin^2 4x}{\tan^2 3x}=\frac{16}{9}$ $\quad[\sin 4x\approx 4x, \tan 3x\approx 3x, \text{ for } x \text{ small}]$ **1**

(b) (i) $AD = 41\cdot 3 \text{ m}$

$\text{arc } DE = 41\cdot 3\times\frac{5\pi}{18}=\frac{413\pi}{36}\text{ m}$ **1**

(ii) $AB = 1\cdot 3$

$\text{arc } AB = 1\cdot 3\times\frac{5\pi}{18}=\frac{13\pi}{36}$

$BE = CD = 40$ m

Area of sector = $\frac{1}{2}r^2\theta$

Shaded area = Sector area ABC – sector area AED

$=\frac{1}{2}(41\cdot 3)^2\frac{5\pi}{18}-\frac{1}{2}(1\cdot 3)^2\frac{5\pi}{18}$ **2**

$=\frac{1}{2}\times\frac{5\pi}{18}\left[(41\cdot 3)^2-(1\cdot 3)^2\right]$

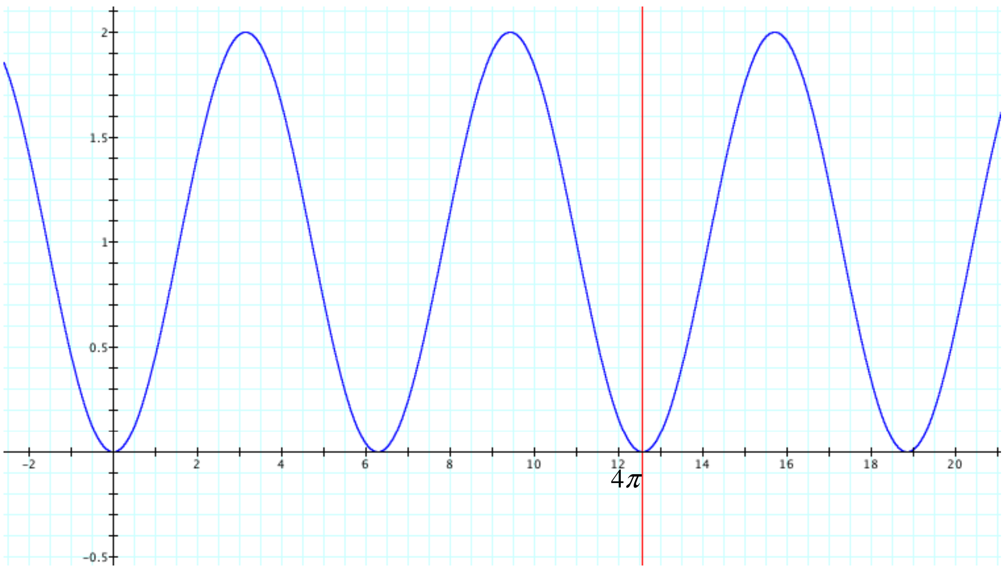
$=\frac{5\pi}{36}\times 1704$

HE7

$=\frac{710\pi}{3}\text{ m}^2$

$\approx 743\cdot 5\text{ m}^2 \quad (1\text{dp})$

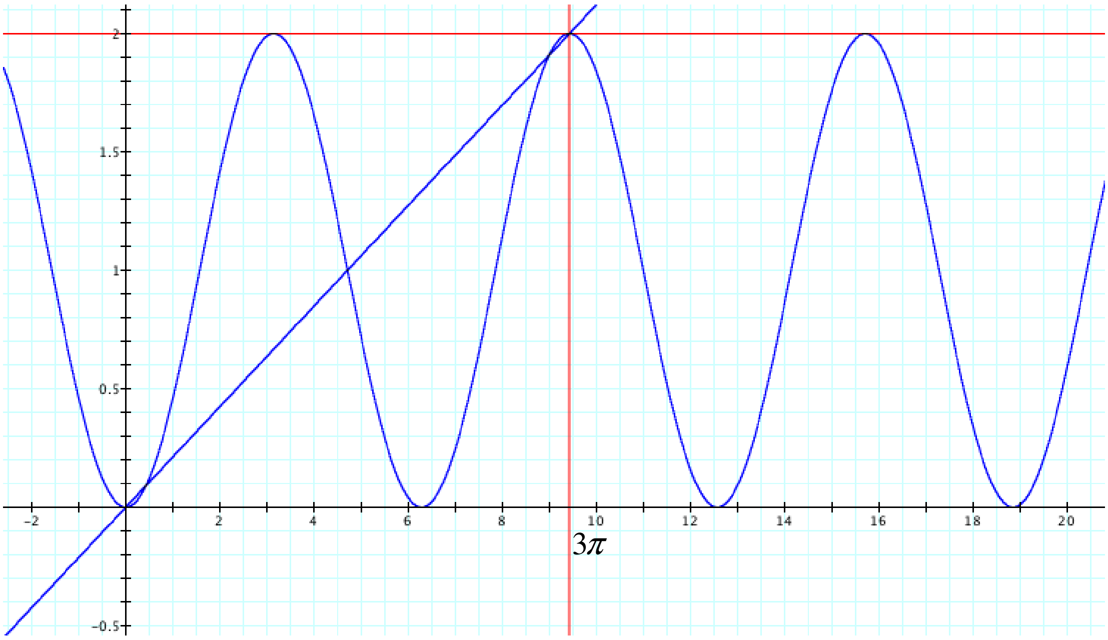
(c) (i) $0\le 1-\cos x\le 2$



3

(c) (ii) $3\pi(1-\cos x) = 2x \Rightarrow 1-\cos x = \dfrac{2x}{3\pi}$

i.e. the intersection of $y = 1-\cos x$ and $y = \left(\dfrac{2}{3\pi}\right)x$



2

So there are 5 solutions for $0 \le x \le 3\pi$.

As $y = \dfrac{2x}{3\pi} \ge 2$ for $x \ge 3\pi$, then there are only 5 solutions for <u>all</u> real x.

Question 5 **Marks**

HE7

(a) (i)
$$\frac{1}{3x-1}-\frac{1}{3x+1}=\frac{3x+1-(3x-1)}{(3x-1)(3x+1)}$$
$$=\frac{2}{9x^2-1}$$
1

(ii)
$$\int\frac{dx}{9x^2-1}=\frac{1}{2}\int\frac{2dx}{9x^2-1}$$
$$=\frac{1}{2}\int\left(\frac{1}{3x-1}-\frac{1}{3x+1}\right)dx \quad [\text{From (i)}]$$
$$=\frac{1}{6}\int\left(\frac{3}{3x-1}-\frac{3}{3x+1}\right)$$
$$=\frac{1}{6}\ln(3x-1)-\frac{1}{6}\ln(3x+1)+c$$
2

PE3

(b)

Method 1:

Two balls drawn for the first game can be done in $\binom{10}{2}=45$ ways.

Both balls are unused means that there are $\binom{4}{2}=6$ ways of doing this.

So the probability of this is $\frac{6}{45}=\frac{2}{15}$. **2**

For the second game, there are still 45 ways to pick two balls, but the two have to be used and so there are $\binom{8}{2}=28$ ways of doing this. So the probability is $\frac{28}{45}$.

So the overall probability is $\frac{2}{15}\times\frac{28}{45}=\frac{56}{675}$

Method 2:

The probability of the first ball not being used is $\frac{4}{10}$ and the next one not being used is $\frac{3}{9}$, so the probability of 2 non-used balls in the first game is $\frac{2}{15}$.

In the second game, the first ball being used is $\frac{8}{10}$ and the second ball being used is $\frac{7}{9}$.

So the probability of the used balls in the second game is $\frac{8}{10}\times\frac{7}{9}=\frac{28}{45}$.

So the probability is $\frac{2}{15}\times\frac{28}{45}=\frac{56}{675}$

(c) (i) $\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)^2=e^x+e^{-x}+2$ **1**

(ii) $V=\pi\int_{-3}^{3}y^2dx=2\pi\int_{0}^{3}y^2dx$

$y^2=\dfrac{e^x+e^{-x}+2}{4}$ $\qquad$ [From (i)]

$$\therefore V=\frac{2\pi}{4}\int_0^3\left(e^x+e^{-x}+2\right)dx$$

$$=\frac{\pi}{2}\left[e^x-e^{-x}+2x\right]_0^3$$ **3**

$$=\frac{\pi}{2}\left[\left(e^3-e^{-3}+6\right)-\left(1-1+0\right)\right]$$

$$=\frac{\pi\left(e^3-e^{-3}+6\right)}{2}$$

So the volume is $\dfrac{\pi\left(e^3-e^{-3}+6\right)}{2}\text{u}^3$

Question 6				**Marks**

HE7

(a) (i) $f(1) = 2\ln 1 - \dfrac{1^2 - 1}{1} = 0$ **1**

(ii) $f(x) = 2\ln x - \dfrac{x^2 - 1}{x} = 2\ln x - x + x^{-1}$

$$\therefore f'(x) = \frac{2}{x} - 1 - x^{-2} = \frac{2}{x} - 1 - \frac{1}{x^2}$$
$$= \frac{2x - x^2 - 1}{x^2} = -\frac{x^2 - 2x + 1}{x^2}$$
$$= -\left(\frac{x-1}{x}\right)^2$$

2

(iii) $f'(x) < 0$ for $x > 0$ and so $f(x)$ is strictly <u>decreasing</u> $(x \neq 1)$ and as $f(1) = 0$ then $f(x) < 0$ for $x > 1$. **1**

Thus $f(x)$ only has one zero i.e. at $x = 1$

(b)
$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x} - 2e^{-2x}}{e^{2x} + e^{-2x}}\,dx$$
$$= \frac{1}{2}\ln\left(e^{2x} + e^{-2x}\right) + C$$

2

HE2

(c) (i)
$$\begin{aligned} \text{LHS} &= (k+2)! \\ &= (k+2)(k+1)! \\ &> (k+2)\times 3^{k-1} && \left[\text{From the fact that } (k+1)! > 3^{k-1}\right] \\ &\geq 3\times 3^{k-1} && [k+2 \geq 3,\ k \text{ positive}] \\ &= 3^k \\ &= \text{RHS} \end{aligned}$$

1

$\therefore (k+2)! > 3^k$

(ii) Test $n = 1$

LHS = 2! = 2

RHS = $3^0 = 1$

So the inequality is true for $n = 1$. **2**

So from (i) and the principle of mathematical induction the inequality is true for all positive integers.

End of solutions