NORTH SYDNEY GIRLS HIGH SCHOOL

2015 **Assessment Task 1**
Term 4

HSC Extension 1 Mathematics

General Instructions

- Reading Time – 2 minutes
- Working Time – 55 minutes
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- In Questions 6–8, show relevant mathematical reasoning and/or calculations

Total Marks – 41 marks

Section I Pages 2–3

5 marks

- Attempt Questions 1–5
- Allow about 7 minutes for this section

Section II Pages 4–7

36 marks

- Attempt Questions 6–8
- Allow about 48 minutes for this section

Student Name:______________________________

Student Number:______________________________

Teacher:

- ○ Mr T
- ○ Ms Lee
- ○ Mr Moon
- ○ Mrs Juhn
- ○ Ms Everingham
- ○ Ms Narayanan
- ○ Mrs Kennedy

QUESTION	PE3	PE4	HE7	TOTAL
1–5	/4		/1	/5
6	/6	/7		/13
7	/6		/6	/12
8	/3		/8	/11
TOTAL	/19	/7	/15	/41

Section I

5 marks
Attempt Questions 1–5
Allow about 7 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5.

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1 A polynomial of degree four is divided by a polynomial of degree two.

What is the maximum possible degree of the remainder?

(A) 3

(B) 2

(C) 1

(D) 0

2 It is known that $(x+3)$ is a factor of the polynomial $P(x)$ and that $P(x) = (x^2+2x+3)Q(x)+(2x+3)$ for some polynomial $Q(x)$.

From this information alone, which of the following can be deduced?

(A) $Q(-3) = 3$

(B) $Q(-3) = -3$

(C) $Q(-3) = \frac{1}{2}$

(D) $Q(-3) = -\frac{1}{2}$

3 Which of the following is the equation of the chord of contact from $(3,-2)$ to the parabola $x^2 = 8y$?

(A) $3x - 4y + 8 = 0$

(B) $3x - 8y + 16 = 0$

(C) $3x - 8y - 8 = 0$

(D) $3x - 4y + 16 = 0$

4 A function is represented by the parametric equations

$$\begin{cases} x = 4 - 2t \\ y = 2t^2 \end{cases}$$

Which of the following is the Cartesian equation of the function?

(A) $y = \frac{1}{2}(4-x)^2$

(B) $y = (4-x)^2$

(C) $y = \frac{1}{4}(4-x)^2$

(D) $y = 4(4-x)^2$

5 Consider the cubic function $y = f(x)$. The graph of the derivative function $y = f'(x)$ cuts the $x-$axis at $(2,0)$ and $(-3,0)$. The maximum value of the derivative is 10.

What is the value of x for which the graph of $y = f(x)$ has a local maximum?

(A) -2

(B) 2

(C) -3

(D) $-\frac{1}{2}$

Section II

36 marks

Attempt Questions 6–8

Allow about 48 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing papers are available.

In Questions 6–8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (13 marks) Use a SEPARATE writing booklet.

(a) If A, B and C are the roots of $x^3 - 8x^2 + 11x + 3 = 0$, evaluate:

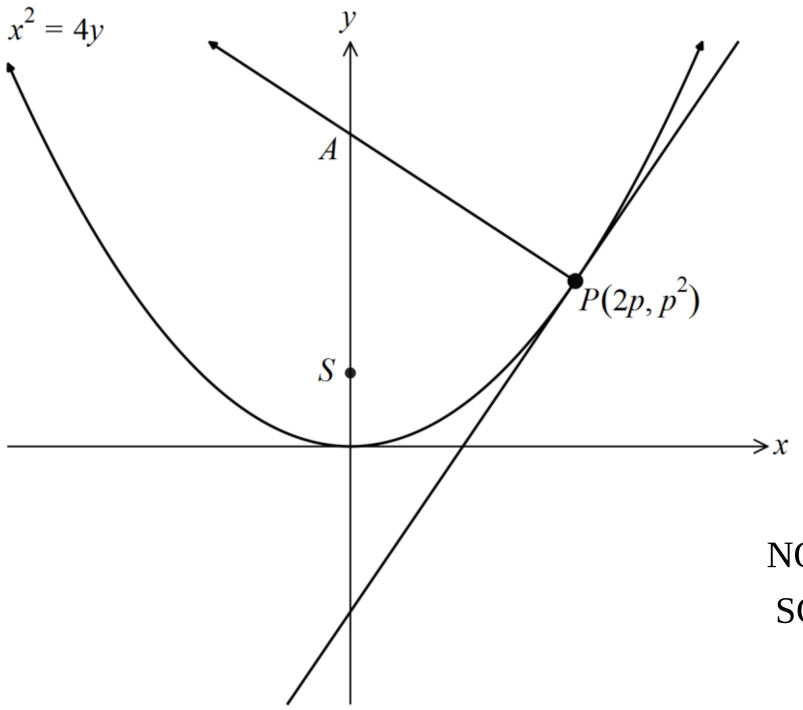
(i) $A + B + C - ABC$ **1**

(ii) $A^2 + B^2 + C^2$ **2**

(b) Find the three roots of the equation $2x^3 - x^2 - 13x - 6 = 0$, given that two of the roots are reciprocals of each other. **3**

Question 6 continues on page 5

(c) $P\left(2p, p^2\right)$ is a variable point on the parabola $x^2 = 4y$, where $p \neq 0$. S is the focus and A is the point where the normal to the parabola at P meets the axis of the parabola.



NOT TO SCALE

(i) Show that the equation of the normal at P is $x + py = p^3 + 2p$. **2**

(ii) Find the coordinates of A. **1**

(iii) M is the midpoint of AP. Show that SM is parallel to the tangent at P. **2**

(iv) The equation of the tangent at P is given by $px - y - p^2 = 0$ and it meets the y-axis at B. Show that the interval AB is bisected by S. **1**

(v) Find the ratio of the areas of ΔASP and ΔABP. **1**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the function $y = f(x)$ where $f(x) = \dfrac{x+1}{4-x^2}$.

(i) Find the equations of all the asymptotes of $y = f(x)$. **2**

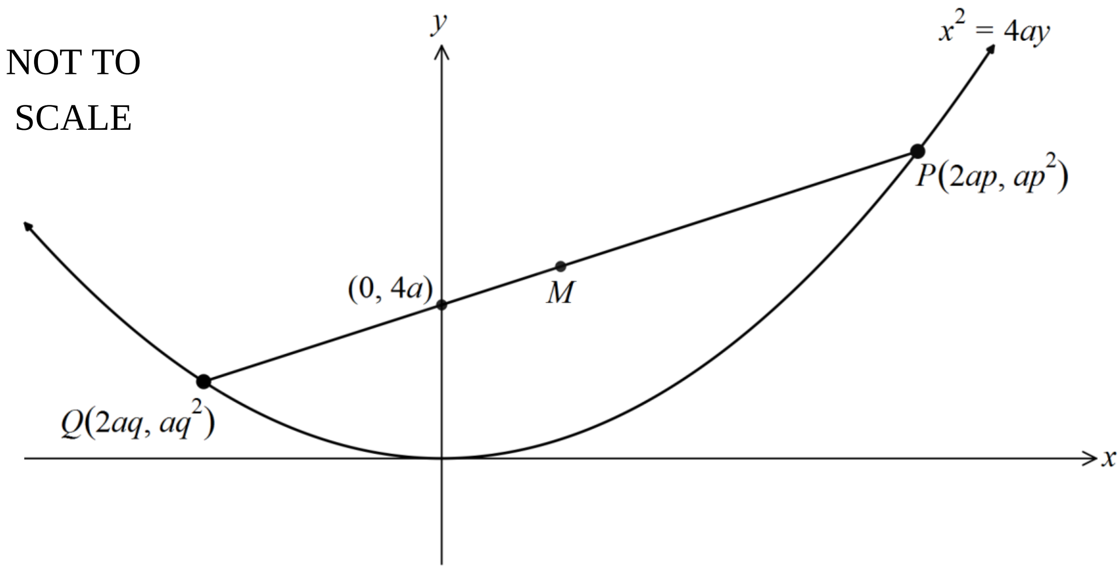
(ii) Show that $y = f(x)$ has no stationary points. **2**

(iii) Sketch $y = f(x)$, showing all intercepts and features. **2**

You are NOT required to find the coordinates of any points of inflexion.

(b) The diagram below shows the graph of the parabola $x^2 = 4ay$.

The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola.



(i) Determine the coordinates of M, the midpoint of the chord PQ. **1**

(ii) The chord PQ passes through the point $(0, 4a)$. Show that $pq = -4$. **3**

(iii) If the chord PQ varies in such a way that it always passes through the point $(0, 4a)$, determine the Cartesian equation of the locus of M. **2**

End of Question 7

Question 8 (11 marks) Use a SEPARATE writing booklet.

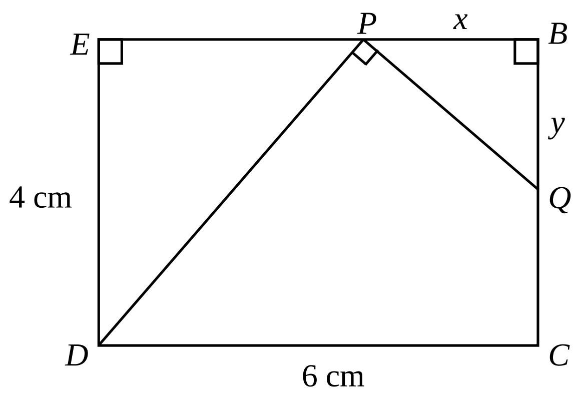
(a) The tangent to the curve $y = x^3 - 2x + 3$ at the point Q meets the curve again at $(-2, -1)$. **3**

Find the coordinates of the point Q.

(b) *EBCD* is a rectangle in which the sides *DC* and *ED* are of length 6 cm and 4 cm respectively.

P is a point on *EB* and *Q* is a point on *BC* such that $\angle DPQ = 90°$.

Let $PB = x$ cm and $BQ = y$ cm.



(i) Explain why $y = \dfrac{x(6 - x)}{4}$. **2**

(ii) Find the value of x for which the area of ΔPBQ is a maximum and find this maximum area. **3**

(iii) It can be shown that the area of ΔDPQ is given by $A = \dfrac{x\left(x^2 - 12x + 52\right)}{8}$. **3**

[**DO NOT PROVE THIS]**

Find the value of x for which this area is a maximum.

End of Paper

2015 Ext1 Mathematics Assessment Task 1 Solutions

Multiple Choice

1 (C)

2 (C)

$$P(-3)=0$$
$$P(-3)=6\times Q(-3)+(-3)$$
$$6Q(-3)=3$$
$$\therefore Q(-3)=\frac{1}{2}$$

3 (A)

$$xx_0=2a(y+y_0)$$
$$3x=4(y-2)$$
$$3x-4y+8=0$$

4 (A)

From $x=2t-4$

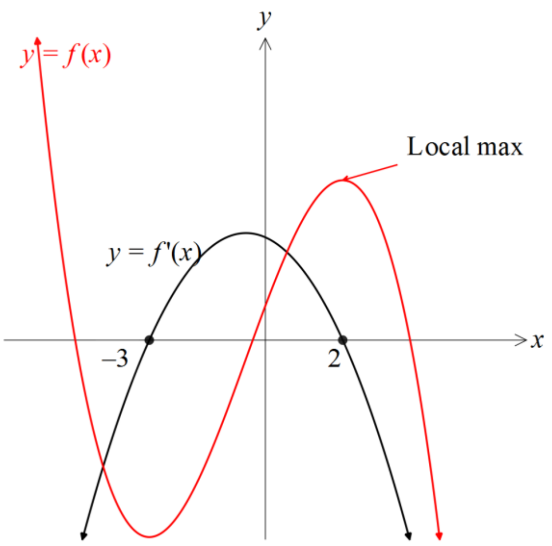
$2t=4-x$

$$\therefore t=2-\frac{x}{2}$$

Sub it into $y=2t^2$

$$y=2\left(2-\frac{x}{2}\right)^2$$
$$=2\left[\frac{1}{2}(4-x)\right]^2$$
$$\therefore y=\frac{1}{2}(4-x)^2$$

5 (B)



Question 6

(a) $x^3-8x^2+11x+3=0$

$A+B+C=8$

$AB+BC+CA=11$

$ABC=-3$

(i) $A+B+C-ABC=8-(-3)=11$

(ii) $A^2+B^2+C^2$

$$=(A+B+C)^2-2(AB+BC+CA)$$
$$=64-22$$
$$=42$$

Marker's Comment

Mostly done very well.

(b) $2x^3-x^2-13x-6=0$

Let $\alpha,\frac{1}{\alpha}$ and β be the roots

Product of roots

$$\alpha\times\frac{1}{\alpha}\times\beta=\frac{6}{2}\qquad\therefore\beta=3$$

Sum of roots

$$\alpha+\frac{1}{\alpha}+\beta=\frac{1}{2}$$
$$\alpha+\frac{1}{\alpha}+3=\frac{1}{2}\quad[\times 2\alpha]$$
$$2\alpha^2+5\alpha+2=0$$
$$(2\alpha+1)(\alpha+2)=0$$
$$\therefore\alpha=-\frac{1}{2},-2$$

Hence the roots are $-2,\ -\frac{1}{2}$ and 3

Marker's Comment

Mostly done very well.

(c) $x^2 = 4y$ $\quad P(2p, p^2)$

(i) $\dfrac{dy}{dx} = 2p \times \dfrac{1}{2} = p$

$\therefore$ the gradient of the tangent at $P(2p, p^2)$ is p.

$\therefore$ the gradient of the normal at $P(2p, p^2)$ is $-\dfrac{1}{p}$.

Hence the equation of the normal at $P(2p, p^2)$ is

$$y - p^2 = -\frac{1}{p}(x - 2p)$$

$$py - p^3 = -x + 2p$$

$$\therefore x + py = p^3 + 2p$$

(ii) A is the y-intercept of the normal

sub $x = 0$

$\therefore 0 + py = p^3 + 2p$

$y = p^2 + 2 \qquad (p \neq 0)$

$\therefore A(0, p^2 + 2)$

(iii) M is the midpoint of $A(0, p^2 + 2)$ and $P(2p, p^2)$.

$\therefore M\left(\dfrac{0 + 2p}{2}, \dfrac{p^2 + 2 + p^2}{2}\right)$

$\therefore M(p, p^2 + 1)$

Now $S(0,1)$ as $a = 1$

$\therefore m_{SM} = \dfrac{p^2 + 1 - 1}{p - 0} = p$

From (i), the gradient of the tangent at P (m_P) is p.

$\therefore m_{SM} = m_P$

Hence two lines are parallel.

(iv) The equation of tangent at P is

$px - y - p^2 = 0$

and B is the y-intercept of the tangent.

Sub $x = 0$, $\therefore y = -p^2$

$\therefore B(0, -p^2)$

$AS = p^2 + 2 - 1 = p^2 + 1$

$BS = 1 + p^2$

$\therefore AS = BS$

Hence AB is bisected by S.

Alternatively

$SM \parallel BP$ and M is the midpoints of AP

(Converse of Join of midpoint theorem)

Hence S is the midpoint of AB.

(v) $AB = 2AS$ (similarity) $\therefore BP = 2SM$

Now

$A_{\Delta ASP} = \dfrac{AP \times SM}{2}$

$A_{\Delta ABP} = \dfrac{AP \times BP}{2}$

$= AP \times SM$

$\therefore A_{\Delta ASP} : A_{\Delta ABP} = 1 : 2$

Marker's Comment

(i) – (iv) was done very well.

(v) It was not done very well. Often with inappropriate approach. The use of similarity was the most effective approach.

Question 7

(a) $f(x)=\dfrac{x+1}{4-x^2}$

(i) $x^2\neq 0 \quad \therefore x=\pm 2$

$\lim\limits_{x\to\infty}\dfrac{x+1}{4-x^2}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}+\frac{1}{x^2}}{\frac{4}{x^2}-1}=0 \qquad \therefore y=0$

Therefore asymptotes are $x=\pm 2,\ y=0$

(ii) $f'(x)=\dfrac{(4-x^2)+2x(x+1)}{(4-x^2)^2}$

$=\dfrac{4-x^2+2x^2+2x}{(4-x^2)^2}$

$=\dfrac{x^2+2x+4}{(4-x^2)^2}$

Now $x^2+2x+4\neq 0$ as $\Delta<0$

$$\left[\begin{aligned}&\Delta=(2)^2-4(1)(4)\\&\quad=-12\\&\therefore \Delta<0\end{aligned}\right]$$

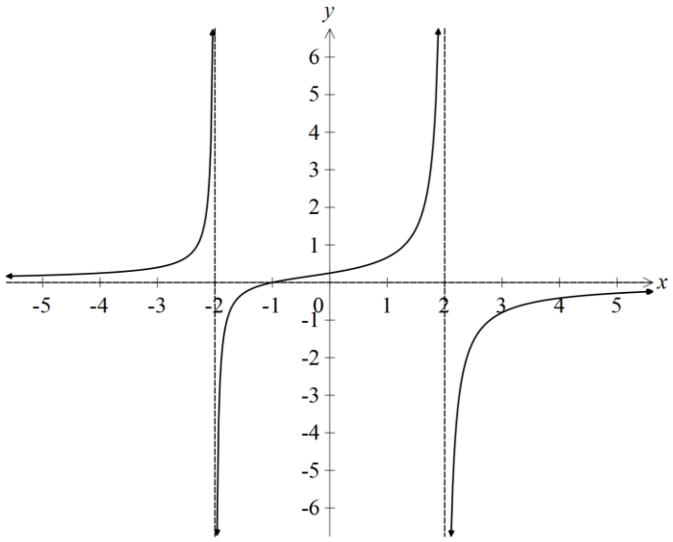
Hence $f'(x)\neq 0$, therefore there is no stationary point.

(iii) x-intercept $=-1$ $\qquad$ y-intercept $=\dfrac{1}{4}$

$x^2+2x+4>0$ and $(4-x^2)^2>0$

$\therefore f'(x)>0$

Hence $f(x)$ is an increasing function.



Marker's Comment

(a) (i) Mostly well done.
Some students forgot to find the horizontal asymptotes.

(ii) Generally well done.
The most common error was factorising x^2+2x+4.
$x^2+2x+4\neq(x+2)^2$ (INCORRECT!)
This was penalised.

Some students did not show why $x^2+2x+4=0$ has no solution.
This was also penalised.

(iii) Many did not draw the graph correctly.

(b) $x^2=4ay$

(i) $M\left(a(p+q),\dfrac{a(p^2+q^2)}{2}\right)$

(ii) The equation of the chord PQ is

$y-ap^2=\dfrac{\cancel{a}\cancel{(p-q)}(p+q)}{2\cancel{a}\cancel{(p-q)}}(x-2ap)$

$y=\dfrac{(p+q)}{2}x-ap(p+q)+ap^2$

$y=\dfrac{(p+q)}{2}x-apq$

Now PQ passes through the point $(0,4a)$.

Sub $(0,4a)$

$4\cancel{a}=-\cancel{a}pq$

$\therefore pq=-4$

Marker's Comment

Mostly done very well.
Some in a variety of ways.

(iii) $M\left(a(p+q),\dfrac{a(p^2+q^2)}{2}\right)$

$$\begin{cases} x=a(p+q) & (1) \\ y=\dfrac{a(p^2+q^2)}{2} & (2) \end{cases}$$

From (1) $(p+q)=\dfrac{x}{a}$

Now

$$p^2+q^2=(p+q)^2-2pq$$
$$=\frac{x^2}{a^2}-2(-4)$$
$$=\frac{x^2}{a^2}+8$$

Sub it into (2)

$$y=\frac{a}{2}\left(\frac{x^2}{a^2}+8\right)$$
$$\therefore y=\frac{x^2}{2a}+4a$$

OR

$$y=\frac{a}{2}\left(\frac{x^2}{a^2}+8\right)$$
$$2ay=x^2+8a^2$$
$$x^2=2ay-8a^2$$
$$x^2=2a(y-4a)$$

Marker's Comment

Generally done well.
Algebra errors only.

Question 8

(a) The tangent to the curve $y=x^3-2x+3$ at the point $Q(\alpha,\beta)$ meets the curve again at $(-2,-1)$. Let the equation of the tangent be $y=mx+b$.
Then to find the point of intersection of the parabola ad the tangent, solve them simultaneously.

$$x^3-2x+3=mx+b$$
$$x^3-(2+m)x+(3-b)=0$$

Roots are α,α and -2 (tangent at $x=\alpha$)
Therefore, sum of roots

$$2\alpha-2=0$$
$$\therefore \alpha=1 \qquad \text{Hence } Q(1,2)$$

Marker's Comment

Some general comments. Number the question parts. It is not the job of the marker to identify which question your answer pertains to. Number your answer booklets eg 1 of 3, 2 of 3 etc. If having multiple attempts at a question, label each attempt clearly with the question number and if you are sure which attempt you wish to be marked, cross out the other attempts by scoring a single line across the working (so it is still legible if the marker needs to look at it).

(a)

This question was not done well. The most common error was finding the equation of the tangent at $(-2,-1)$. This earned you only ½ a mark out of the available 3.

Draw a diagram. This will help you to clarify the information given in the question.
The gradient of the tangent is the same as the gradient of the curve at the point of contact. So you need to use the coordinates of the point of contact to find the gradient of the tangent. If you don't know this point use (x_1,y_1) or similar.

Some students assumed a double root for the original polynomial. The roots of the equation correspond to intercepts of the original polynomial not the intersections of the tangent and the curve. This earned you only 1 mark out of 3.

The solutions show two alternate approaches both of which were successfully used by several students.

(b)(i) $EP = 6 - x$, $CQ = 4 - y$

$DQ^2 = 36 + (4 - y)^2$

$DP^2 = 16 + (6 - x)^2$

$PQ^2 = x^2 + y^2$

But $DQ^2 = DP^2 + PQ^2$

$36 + (4 - y)^2 = 16 + (6 - x)^2 + x^2 + y^2$

$20 + 16 - 8y + y^2 = 36 - 12x + x^2 + x^2 + y^2$

$8y = 12x - 2x^2$

$$y = \frac{6x - x^2}{4} = \frac{x(6 - x)}{4}$$

Markers Comments

(b) (i)

Similarity arguments and Pythagoras' Theorem both could be used to establish the desired result. For those using the Pythagoras' approach, remember it is easier to work with squares of distances than with square roots.

Many students embarked on a full page geometric argument to establish similarity. This was not asked for – read the key words in the question "Explain why" versus "Prove that" and also note the number of marks allocated. On the other end of the spectrum, some students provided no detail at all. Whilst a full similarity proof was not required, adequate detail needed to be included to clearly communicate how the result is arrived at.

(ii) Area ΔPBQ,

$$A = \frac{xy}{2} = \frac{x}{2} \times \frac{6x - x^2}{4} = \frac{6x^2 - x^3}{8}$$

$$\frac{dA}{dx} = \frac{1}{8}\left(12x - 3x^2\right) = 0 \text{ when } 3x(4 - x) = 0,$$

ie. when $x = 0, 4$

$$\frac{d^2A}{dx^2} = \frac{1}{8}\left(12 - 6x\right).$$

When $x = 4$, $\frac{d^2A}{dx^2} = \frac{-12}{8} < 0$

Maximum area when $x = 4$.

$$\text{Area} = \frac{6 \times 4^2 - 4^3}{8} = 4 \text{ cm}^2$$

Markers Comments

(ii)

A large number of students differentiated *y*. Unfortunately this meant that we could not award you marks as you answered a completely different question. Also, to optimise *y*, which is a parabola, you do not need to use the heavy-lifting tools of calculus. In optimisation questions, always first note what is to be optimised and then develop an expression for it.

Some students realised they needed the area but decided to call their Area function *y*. There is already a *y* in the question. Use a different pronumeral. There are 24 other letters to choose from!

Many students solved for $A' = 0$ without ever communicating what they are doing or why. This was penalised. Many others wrote that $A' = 0$ for a maximum (which is not always the case) but incurred no penalty.

It is not enough to find the stationary points, you must test the nature to establish you have a point of maximum.

Some students forgot to evaluate the maximum area once they found where the maximum occurred.

A note on differentiation. If you are tempted to use the quotient rule, ask yourself if the denominator is variable or a constant. You do not need quotient rule if the denominator is a not a variable expression.

(iii) Area $\Delta DPQ = \dfrac{x\left(x^2 - 12x + 52\right)}{8}$

Call area H, $H = \dfrac{x^3 - 12x^2 + 52x}{8}$

$$\frac{dH}{dx} = \frac{1}{8}\left(52 - 24x + 3x^2\right)$$

Now, $3x^2 - 24x + 52 \neq 0$ as

$\Delta = 24^2 - 4 \times 3 \times 52 = -48 < 0$

$\therefore$ $3x^2 - 24x + 52 = 0$ has no real roots.

$\therefore \dfrac{dH}{dx} \neq 0$

$\therefore$ H has no stationary points

$x = 0$, Area $\Delta DPQ = 0$

$x = 6$, Area $\Delta DPQ = \dfrac{6(36 - 72 + 52)}{8} = 12$

The greatest area occurs at one of the endpoints of the domain, $x = 6$ when DQ is a diagonal of the rectangle and $\angle DPQ$ coincides with $\angle DEB$.

Markers Comments

(iii) Most students could differentiate and set $A' = 0$.
However, the majority of students were confused by the fact that the equation yielded no solutions. If you wrote in your working that you were solving $A' = 0$ in order to find stationary points, perhaps it would have been clearer that no solutions simply means there are no stationary points. Check the end-points of the domain.