Name: ....................................

Teacher: ..................................

SCEGGS Darlinghurst

## HSC Assessment 2
## Tuesday, 6th June, 2006

# Extension 1 Mathematics

### General Instructions

- Time allowed – 75 minutes
- Weighting 35%
- This paper has **four** questions
- Attempt **all** questions and show all necessary working
- Marks may be deducted for careless or badly arranged work
- Write using blue or black pen, diagrams in pencil
- Write your name and your teacher's name at the top of each page
- Approved calculators, mathematical templates and geometrical instruments may be used
- A table of standard integrals is provided at the back of this paper

| Questions | Total | Comm. | Reas. | Calc. |
|---|---|---|---|---|
| 1 | /12 | /2 | /1 | /5 |
| 2 | /10 | | /2 | /8 |
| 3 | /12 | /2 | /3 | /1 |
| 4 | /13 | /1 | /6 | /4 |
| **TOTAL** | **/47** | **/5** | **/12** | **/18** |

**Question 1** (12 marks) **Marks**

(a) Find the inverse function of $y = 3 + \log_e x$ 1

(b) Differentiate $\cos^{-1}(2x)$ 1

(c) Evaluate $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{4+x^2}$ 2

(d) Find $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}}$ 2

(e) Consider the function $f(x) = x^2 - 2x$

i. Sketch $y = f(x)$ clearly indicating the coordinates of the vertex and any intercepts. Use the same scale on both axes. 1

ii. State the largest domain that includes the origin for which the function has an inverse function. 1

iii. State the domain and range of this inverse function, $f^{-1}(x)$. 2

iv. Sketch $y = f^{-1}(x)$ on the same set of axes as in part(i). Label the two graphs clearly. 1

v. Find the gradient of $y = f^{-1}(x)$ at the origin. 1

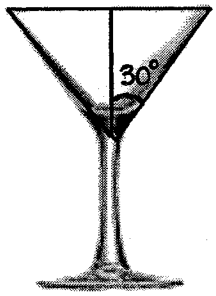
**Question 2 begins on page 2 ...**

START A NEW PAGE

**Question 2** (10 marks) **Marks**

(a) Using the substitution $u = x + 1$, evaluate $\int_1^3 \frac{x-1}{(x+1)^3}\,dx$ 4

(b) A cocktail glass is in the shape of an inverted right-cone with semi-vertical angle 30°. It is initially filled with liquid to a height $h$ cm.




i. Show that the volume of liquid in the glass is given by: 2

$$V = \frac{1}{9}\pi h^3$$

ii. The liquid is drunk (through a straw) at a rate of 1 mL/s. Find the rate at which the height of the liquid is changing when there is only 10 mL of liquid left. Answer to 2 decimal places. 4
[Note: $1\,\text{cm}^3 = 1\,\text{mL}$]

**Question 3 begins on page 3 . . .**

START A NEW PAGE

**Question 3** (12 marks) **Marks**

(a) Find the exact value of 3

$$\cos\left(\sin^{-1}\left(\frac{1}{3}\right)+\cos^{-1}\left(-\frac{2}{5}\right)\right)$$

(b) A hot cup of coffee loses heat in a colder environment according to Newton's Law of Cooling, $\frac{dT}{dt} = -k(T - T_e)$, where $t$ is time in minutes, $k$ is a constant, and $T_e$ is the temperature of the environment.

i. Show that $T = T_e + Ae^{-kt}$ is a solution of this equation, for some constant $A$. 1

At 7 am on a cold morning (10°C) I buy a cup of coffee. The coffee cools from 75°C to 65°C in 15 minutes.

ii. Find the exact value of the constants $A$ and $k$. [Note that $T_e = 10$] 2

iii. What is the temperature of the coffee when I arrive at work at 7:20 am? (Answer to the nearest degree) 1

iv. How long do I have to drink my coffee before the temperature falls below 55°C and is too cold to drink? (Answer to the nearest minute) 1

v. At what rate was the tea cooling initially? 2

vi. Sketch a graph of the temperature of the coffee against time. Label any important features. 2

**Question 4 begins on page 4 . . .**

**START A NEW PAGE**

**Question 4** (13 marks) **Marks**

(a) Consider the function $f(x) = 3\cos^{-1}\frac{x}{2}$

i. State the domain and range of $y = f(x)$ 2

ii. Neatly sketch $y = f(x)$ 1

iii. Find the area of the region in the first quadrant bounded by the curve $y = f(x)$ and the co-ordinate axes. 3

(b) i. Using the substitution $u = 1 - x^2$, or otherwise, find 3

$$\int \frac{x}{\sqrt{1-x^2}}\,dx$$

ii. Differentiate $x\sin^{-1}x$ 1

iii. Hence find 3

$$\int_0^{\frac{1}{2}} \sin^{-1}x\,dx$$

**END OF ASSESSMENT**

# STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Reas. /1
Calc. /5
Comm. /2

**Question 1** (12 marks)

(a) Original: $y = 3 + \log_e x$

Inverse: $x = 3 + \log_e y$

$x - 3 = \log_e y$

$y = e^{x-3}$ ✓

Once swap $x$ & $y$, must make $y$ subject

(b) $y = \cos^{-1} 2x$

$y' = \dfrac{-2}{\sqrt{1-4x^2}}$ ✓ calc.

Lots of people forgot
- chain rule
- negative

(c) $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{4+x^2}$

$= \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^{2\sqrt{3}}$ ✓

$= \frac{1}{2}\tan^{-1}\sqrt{3} - \frac{1}{2}\tan^{-1} 0$

$= \frac{\pi}{6}$ ✓ calc.

- Lots of people forgot the $\frac{1}{2}$ out front
- Must evaluate $\tan^{-1}\sqrt{3}$ exactly. Did not accept a rounded decimal.

(d) $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}}$

$= \displaystyle\int \frac{dx}{\sqrt{3^2-(2x)^2}}$

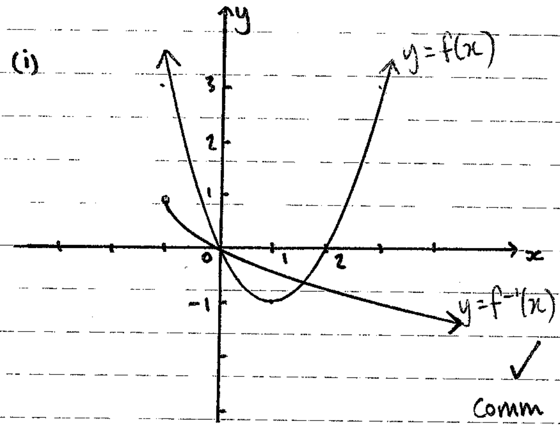
$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$ calc.

✓ ✓

- People forgot the $\frac{1}{2}$ (Reverse Chain Rule)

(e) $f(x) = x^2 - 2x$

(i)




✓ Comm

- The inverse needs to go through (3, −1) if it was well off you didn't get the mark
- Shouldn't need a page of calculus to draw a parabola

(ii) $x \le 1$ ✓

Generally well done

(iii) $f^{-1}(x)$: D: $x \ge -1$, $x$ real ✓
R: $y \le 1$, $y$ real ✓

(iv) (shown above) ✓ comm.

(v) original: $y = x^2 - 2x$

inverse: $x = y^2 - 2y$

$\frac{dx}{dy} = 2y - 2$

$\frac{dy}{dx} = \frac{1}{2y-2}$

@ (0,0) $m_T = \frac{1}{2\times 0 - 2}$

$= -\frac{1}{2}$ ✓ Reas

People either knew it or they didn't.

## Question 2

Reas. /2
Calc. /8 (10 marks)

(a) $\int_1^3 \frac{x-1}{(x+1)^3}\,dx$  $u = x+1$, $du = dx$

$= \int_2^4 \frac{u-2}{u^3}\,du$ ✓  $x=1 \rightarrow u=2$, $x=3 \rightarrow u=4$ ✓

$= \int_2^4 u^{-2} - 2u^{-3}\,du$

$= \left[\frac{u^{-1}}{-1} - \frac{2u^{-2}}{-2}\right]_2^4$ ✓

$= \left[\frac{-1}{u} + \frac{1}{u^2}\right]_2^4$

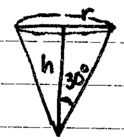
$= \left(\frac{-1}{4} + \frac{1}{16}\right) - \left(\frac{-1}{2} + \frac{1}{4}\right)$

$= \frac{1}{16}$ ✓ (Calc4.)

This part was well done.

You will not be given any marks for integrating incorrectly and substituting into nonsense.

(b)(i)




$V_{cone} = \frac{1}{3}\pi r^2 h$

$\tan 30° = \frac{r}{h}$

$r = h\tan 30°$

$r = \frac{h}{\sqrt{3}}$ ✓

$\Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$

$\Rightarrow V = \frac{\pi h^3}{9}$ ✓ (Reas/2)

Clearly show this expression for the radius. AND Substitute it into the volume of a cone formula.

(ii) $\frac{dV}{dt} = 1 \frac{mL}{s}$, $\frac{dh}{dt} = ?$ when $V = 10$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

↓ find this!

$V = \frac{1}{9}\pi h^3$

$\frac{dV}{dh} = \frac{1}{3}\pi h^2$

$\frac{dh}{dV} = \frac{3}{\pi h^2}$ ✓

$\therefore \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{3}{\pi h^2} \times 1$

$= \frac{3}{\pi h^2}$ ✓

This part was well done.

When $V = 10$

$\frac{1}{9}\pi h^3 = 10$

$h = \sqrt[3]{\frac{90}{\pi}}$ ✓

$\therefore \frac{dh}{dt} = \frac{3}{\pi\left(\sqrt[3]{\frac{90}{\pi}}\right)^2} = 0.10199\ldots$

$= 0.10$ to 2 d.p.

$\therefore$ The height of the liquid is changing at 0.10 cm/s ✓ (Calc4)

Many students forgot that 10 was volume and not the height.
Read the units very carefully.
10 mL = 10 cm³
∴ It is a volume not a height.

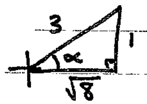
Calc /1
Comm /2
Reas /3

## Question 3 (12 marks)

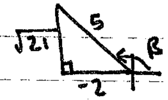
(a) $\cos(\sin^{-1}\frac{1}{3} + \cos^{-1}-\frac{2}{5})$

$\downarrow \alpha$ $\qquad \downarrow \beta$

let $\alpha = \sin^{-1}\frac{1}{3}$




let $\beta = \cos^{-1}-\frac{2}{5}$




$\therefore \cos(\sin^{-1}\frac{1}{3} + \cos^{-1}-\frac{2}{5})$

$= \cos(\alpha+\beta)$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$ ✓

$= \frac{\sqrt{8}}{3} \times \frac{-2}{5} - \frac{1}{3} \times \frac{\sqrt{21}}{5}$ ✓ ✓

$= \frac{-2\sqrt{8} - \sqrt{21}}{15}$ Reas.

ok, except for some careless errors two people had $\sqrt{21}$ turn into $\sqrt{2}$ for example!

These questions are easy to check by using your calculator! No one should be losing marks for carelessness in these type of questions.

(b)(i) $T = T_e + Ae^{-kt}$

$\frac{dT}{dt} = A \times e^{-kt} \times -k$

$= -k \times (Ae^{-kt})$

$= -k(T - T_e)$ ✓ Calc

Well done.

(ii) $t = 0 \quad T = 75°C$

$\Rightarrow 75 = 10 + Ae^{-k\times 0}$

$A = 65$ ✓

$t = 15 \quad T = 65°C$

$\Rightarrow 65 = 10 + 65e^{-k\times 15}$

$55 = 65e^{-15k}$

$k = \frac{-1}{15}\log_e\left(\frac{55}{65}\right)$ ✓

$(k \doteqdot 0.0111369\ldots)$

(iii) $T = 10 + 65e^{-kt}$

$t = 20 \quad T = ?$

$T = 10 + 65 \times e^{-k\times 20}$

$= 62.021\ldots$

$\doteqdot 62°$ (to nearest degree) ✓

(iv) $T = 55 \quad t = ?$

$55 = 10 + 65e^{-kt}$

$45 = 65e^{-kt}$

$t = \frac{-1}{k}\log_e\left(\frac{45}{65}\right)$

$= 33.018\ldots$

$\doteqdot 33$ minutes ✓

Falling below 55°C does not mean solve for 54°! (54.9°C is still below!)

(v) $t = 0 \quad \frac{dT}{dt} = ?$

$\downarrow$

$T = 75$

$\frac{dT}{dt} = -k(T - T_e)$ ✓

$= -k(75 - 10)$

$= -65k$

$= -0.7239\ldots$

$\doteqdot -0.724$ °c/min ✓

or/ $\frac{dT}{dt} = -65ke^{-kt}$

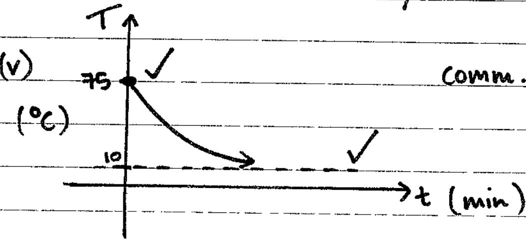
$= -65k$ when $t = 0$

$\doteqdot -0.724$ °c/min

Make sure you include units!!!

(& apologies for the typo)

(v)




Comm.

Lumpy graphs did not get full marks. Plotting points is fine (& good) but make sure they can be joined with a smooth curve & the scale allows it to approach the asymptote.

## Question 4

comm. /1
Reas. /6
Calc. /4
(13 marks)

(a) $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right)$

(i) $\left(\frac{y}{3}\right) = \cos^{-1}\left(\frac{x}{2}\right)$

R: $0 \le \frac{y}{3} \le \pi$

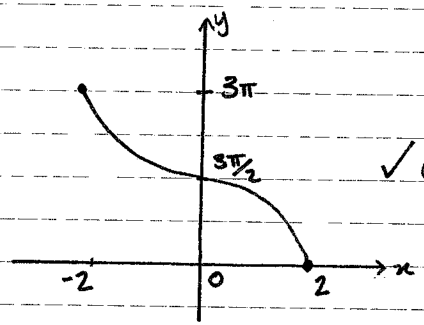
R: $0 \le y \le 3\pi$ ✓

D: $-1 \le \frac{x}{2} \le 1$

D: $-2 \le x \le 2$ ✓

Well done!

(ii)




✓ Comm.

The value of the y intercept at $y = 3\pi/2$ is very important.

(iii) $y = 3\cos^{-1}\left(\frac{x}{2}\right)$

$\left(\frac{y}{3}\right) = \cos^{-1}\left(\frac{x}{2}\right)$

$x = 2\cos\left(\frac{y}{3}\right)$

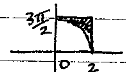
$\text{Area} = \int_0^{3\pi/2} 2\cos\frac{y}{3}\,dy$ ✓

$= \left[\frac{2\sin\left(\frac{y}{3}\right)}{1/3}\right]_0^{3\pi/2}$ ✓

$= \left[6\sin(y/3)\right]_0^{3\pi/2}$

$= 6\sin(\pi/2) - 6\sin(0)$

$= 6\ u^2$ ✓ Reas.




Note: Rectangle $- \int 2\cos y/3$ gives the wrong part.

Make sure you divide the $\frac{1}{3}$ not multiply by it.

(b) (i) $\int \frac{x}{\sqrt{1-x^2}}\,dx$  $u = 1 - x^2$, $du = -2x\,dx$, $-\frac{1}{2}du = x\,dx$

$= -\frac{1}{2}\int \frac{du}{\sqrt{u}}$ ✓

$= -\frac{1}{2}\int u^{-1/2}\,du$

$= -\frac{1}{2} \times \frac{u^{1/2}}{1/2} + C$ ✓

$= -\sqrt{u} + C$

$= -\sqrt{1-x^2} + C$ ✓ Calc.

• do not mix up the letters in your substitution lines.

The final answer must be written in terms of $x$.

Note $\sqrt{1-x^2} \ne 1-x$ Never, ever, ever!

(ii) $\frac{d}{dx}\left(x\sin^{-1}x\right)$ Use the product rule

$= x \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x$

$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$ ✓ Calc.

product rule $uv' + vu'$

(iii) $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x\,dx = \left[x\sin^{-1}x\right]_0^{1/2}$

$\int_0^{1/2} \sin^{-1}x\,dx = \left[x\sin^{-1}x\right]_0^{1/2} - \int_0^{1/2}\frac{x}{\sqrt{1-x^2}}\,dx$ ✓

$= \left[x\sin^{-1}x\right]_0^{1/2} + \left[\sqrt{1-x^2}\right]_0^{1/2}$ ✓

$= \left[\frac{1}{2}\sin^{-1}\frac{1}{2} - 0\right] + \left[\sqrt{1-1/4} - \sqrt{1-0}\right]$

$= \frac{1}{2} \times \frac{\pi}{6} + \sqrt{3/4} - 1$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$ ✓

Reas.

this is Reverse Product Rule.

The question states hence

So you must use part (i) and (ii)

if the question stated or otherwise then you could use a different method.