Centre Number

Student Number

SCEGGS Darlinghurst

2007
Higher School Certificate
Assessment Task 2

Mathematics-Extension I

Task Weighting: 35%

Outcomes Assessed: **HE3, HE4, HE6 & HE7**

General Instructions

- Time allowed – 60 minutes
- **Start each question on a new page.**
- Attempt **all** questions and show all necessary working.
- Answer **Question 1 (c)** on the answer sheet provided
- Write your student number at the top of each page.
- Marks can be deducted for careless or badly arranged work
- Mathematical templates, geometrical equipment and scientific calculators may be used.

Question	Reasoning	Communication	Total	Calc
1	/2	/4	/12	/6
2	/7	/1	/13	
3	/3	/3	/13	/3
4	/7	/1	/12	/2
Total	/19	/9	/50	/11

Average:__________ **St. Dev.:**__________ **Rank:**__________

Parent's Signature____________________

Start A New Page

Marks

Question 1: (12 marks)

(a) Find:

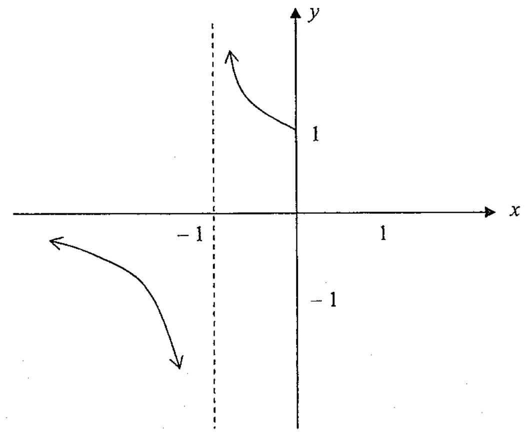
(i) $\displaystyle\int \frac{dx}{\sqrt{16-x^2}}$ **2**

(ii) $\displaystyle\int x(x-2)^5\,dx$ using the substitution $u = x-2$ **2**

(b) Evaluate:

$$\int_{-\frac{1}{4}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-4x^2}}$$ **2**

(c) The graph of $y = f(x)$ is drawn below: **2**



On the answer sheet provided sketch $y = f^{-1}(x)$

Question 1 continued on the next page

Marks

Question 1 (continued)

(d) The half life of a radioactive substance is the time it takes a given amount of the substance to lose one half of its mass. It is given that the half-life of plutonium-239 is 24 000 years. **2**

Assume that plutonium-239 decays according to the law:

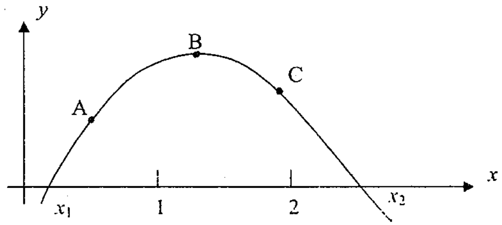
$M = M_0 e^{-kt}$ where M = mass

t = time in years

M_0 and k are constants

Find how long it would take for an amount of plutonium-239 to lose 80% of its mass. (Answer to nearest year)

(e) Consider the graph of $y = f(x)$



Siobhan wants to calculate the value of x_1. It is the root of the equation $f(x) = 0$ between 0 and 1. She uses Newton's Method.

(i) Explain, graphically, why the x – value of A is a better choice than the x – value of C as a first approximation of x_1. **1**

(ii) Explain what would happen if Siobhan used the x – value of B. **1**

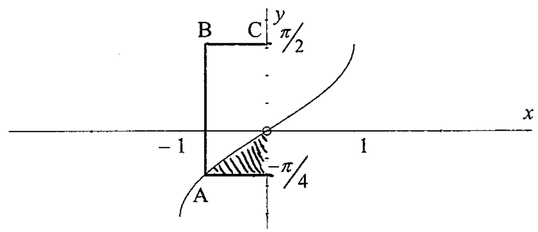
Start A New Page

Marks

Question 2: (13 marks)

(a) $x^4 - 10x + 7 = 0$ has a root between 0.6 and 0.9. Use halving the interval method twice to show the root lies between 0.675 and 0.75. **2**

(b) The diagram shows a sketch of the function $y = \sin^{-1} x$



(i) What are the coordinates of B? **1**

(ii) Show the area of the shaded region is $\dfrac{2-\sqrt{2}}{2}$ units^2. **3**

(iii) Hence calculate the area bounded by $y = \sin^{-1} x$, the y – axis and the intervals AB and BC **1**

(c) An ice cube tray is filled with water at a temperature of 18^0C and placed in a freezer that has a constant temperature of -19^0C. The cooling rate of the water is proportional to the difference between the temperature of the freezer and the temperature of the water T.

T satisfies the equation $\dfrac{dT}{dt} = k(T + 19)$

(i) Show that $T = -19 + Ae^{kt}$ satisfies the equation for $\dfrac{dT}{dt}$ and find the value of A. **2**

(ii) After 5 minutes in the freezer the temperature of the water 3^0C. Find the time for the water to reach -18.9^0C. **3**

(iii) Sketch a graph of Temperature versus Time labelling all important features. **1**

Start A New Page

Marks

Question 3: (13 marks)

(a) (i) Show that the equation $e^x = x+2$ has a solution in the interval $1 < x < 2$. **1**

(ii) Letting $x_1 = 1.5$ use one application of Newton's Method to approximate the solution to 3 decimal places. **3**

(b) By using the substitution $x = \tan\theta$ evaluate $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{\left(1+x^2\right)^{3/2}}$ **3**

(c) Consider the function $y = \cos^{-1}(x-1)$.

(i) Find the domain of the function. **1**

(ii) Sketch the graph of the curve $y = f(x)$ showing clearly the coordinates of the endpoints. **2**

(iii) The region in the first quadrant bounded by the curve $y = f(x)$ and the coordinate axes is rotated about the y – axis. Find the exact value of the volume of the solid or revolution. **3**

Start A New Page

Marks

Question 4: (12 marks)

(a) Find the exact value of $\sin\left[\tan^{-1}\left(\frac{-1}{2}\right) + \cos^{-1}\left(\frac{2}{3}\right)\right]$ **3**

(b) (i) Show that $\frac{u}{u+1} = 1 - \frac{1}{u+1}$ **1**

(ii) Hence find $\int \frac{dx}{1+\sqrt{x}}$ using the substitution $x = u^2$ **2**

(c) $y = \sinh x$ is an example of a hyperbolic function. It is defined as:

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

(i) Find $\frac{dy}{dx}$ **1**

(ii) Explain why $y = \sinh x$ has an inverse function **1**

(iii) Show that $f^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$ for all x. **4**

END OF PAPER

Start A New Page

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Mathematics Extension 1 – Assessment Task 2 HSC 2007 – Solutions

/2 a) i) $\displaystyle\int \frac{dx}{\sqrt{16-x^2}} = \sin^{-1}\frac{x}{4} + c$ ✓✓ Calc - 2

/2 ii) $\displaystyle\int x(x-2)^5\,dx$ let $u = x-2$, $du = 1.dx$

$= \displaystyle\int (u+2)u^5\,du$ ✓

$= \displaystyle\int u^6 + 2u^5\,du$

$= \dfrac{u^7}{7} + \dfrac{u^6}{3} + c$

$= \dfrac{(x-2)^7}{7} + \dfrac{(x-2)^6}{3} + c$ ✓ Calc – 2

Remember to change back to x!

/2 b) $\displaystyle\int_{-1/4}^{1/2} \frac{dx}{\sqrt{1-4x^2}}$

$= \dfrac{1}{2}\displaystyle\int_{-1/4}^{1/2} \frac{dx}{\sqrt{\frac{1}{4}-x^2}}$

$= \dfrac{1}{2}\left[\sin^{-1} 2x\right]_{-\frac{1}{4}}^{\frac{1}{2}}$ ✓

$= \dfrac{1}{2}\left(\sin^{-1}1 - \sin^{-1}\left(-\tfrac{1}{2}\right)\right)$

$= \dfrac{1}{2}\left(\dfrac{\pi}{2} - -\dfrac{\pi}{6}\right)$

$= \dfrac{\pi}{3}$ ✓ Calc – 2

/2 c) Refer to answer sheet ✓✓

Intersection must be on the line $y = x$.

Communication – 2

/2 d) find k if $M = \frac{M_0}{2}$ when $t = 24\,000$

$$\therefore \frac{M_0}{2} = M_0 e^{k \times 24000}$$
$$\frac{1}{2} = e^{k \times 24000}$$
$$\ln\frac{1}{2} = 24000 \times k$$
$$k = \frac{\ln(\frac{1}{2})}{24000}$$ ✓
$$= -2.89 \times 10^{-5} \quad (3 \text{ sig fig})$$

find t when $M = 0.2M_0$

$$\therefore 0.2M_0 = M_0 e^{kt}$$
$$0.2 = e^{kt}$$ Reasoning - 2
$$\ln(0.2) = kt$$
$$t = \frac{\ln(0.2)}{k}$$
$$= \frac{\ln(0.2)}{\ln(\frac{1}{2})} \times 24\,000$$
$$= 55\,726 \quad (\text{to nearest whole no.})$$ ✓

$\therefore$ It will take 55 726 years to lose 80% of mass

note: $M \neq 0.8M_0$!!

/1 e) i) the tangent at A cuts the x-axis closer to x_1 than the tangent at C. ✓

Communication - 1

draw a diagram must mention the tangent and that the new approximation is where the tangent crosses the x axis.

/1 a) the gradient of the tangent is zero

$\therefore f'(x) = 0 \quad \therefore \frac{f(x)}{f'(x)}$ is undefined and

Newton's Method does not work. ✓

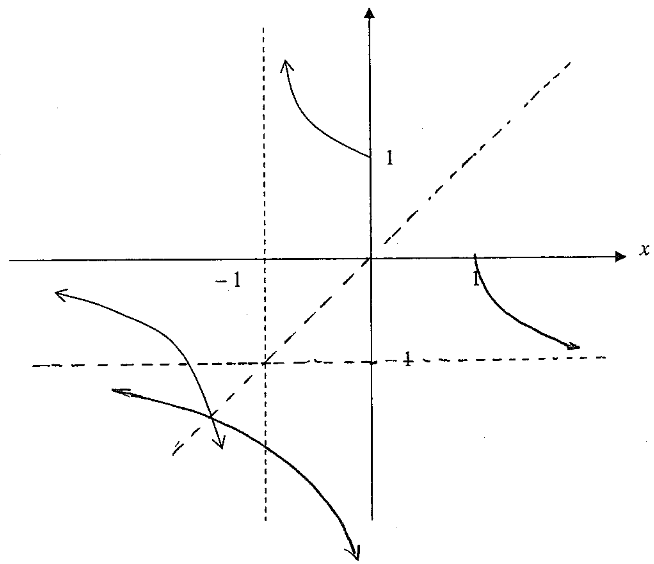
or

The tangent does not cut the x-axis.

Communication - 1

Start A New Page

Answer Sheet for Question 1 (c)



Q2 a) $f(x) = x^4 - 10x + 7$

/2

$f(0.6) = 1.1296$

$f(0.9) = -1.3439$

$\frac{0.6+0.9}{2} = 0.75$

$f(0.75) = -0.18359\ldots$ ✓

$\therefore$ root lies between 0.6 and 0.75

$\frac{0.6+0.75}{2} = 0.675$

$f(0.675) = 0.45759\ldots$ ✓

$\therefore$ root lies between 0.675 and 0.75

To draw conclusions you must state the value of f(0.6) & f(0.9)

because it is a 'show' question you must state the value of f(0.75) & f(0.675)

/1 b) i) $B\left(-\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right)$ ✓

done well

/3 ii) $y = \sin^{-1} x$

$x = \sin y$

$\text{Area} = \left|\int_{-\frac{\pi}{4}}^{0} \sin y \, dy\right|$ ✓

$= \left|\left[-\cos y\right]_{-\frac{\pi}{4}}^{0}\right|$

$= \left|(-\cos 0) - \left(-\cos\left(-\frac{\pi}{4}\right)\right)\right|$ ✓

$= \left|-1 - \left(-\frac{1}{\sqrt{2}}\right)\right|$

$= \left|\frac{1}{\sqrt{2}} - 1\right|$

$= \left|\frac{1-\sqrt{2}}{\sqrt{2}}\right|$

$= \left|\frac{\sqrt{2}-2}{2}\right|$ ✓

$= \frac{2-\sqrt{2}}{2} \text{ units}^2$

Reasoning - 3

some interesting statements here!

because it is a show question you must be particular about how you find this area

eg's why does these work:

$\cdot \int_0^{\pi/4} \sin y \, dy$

$\cdot -\int_0^{\pi/4} \sin y \, dy.$

etc

/1 iii) $\text{Area} = \frac{1}{\sqrt{2}} \times \frac{3\pi}{4} - \left(\frac{2-\sqrt{2}}{2}\right)$ ✓ Reasoning - 1

$= \frac{3\sqrt{2}\pi}{8} - \left(\frac{2-\sqrt{2}}{2}\right)$

$= \frac{1}{8}\left(3\sqrt{2}\pi + 4\sqrt{2} - 8\right) \text{ units}^2$

c) i) $T = -19 + Ae^{kt}$

/2

$\frac{dT}{dt} = kAe^{kt}$

$\therefore \text{LHS} = \frac{dT}{dt}$

$= kAe^{kt}$

$\text{RHS} = k(T+19)$

$= k\left(-19 + Ae^{kt} + 19\right)$ ✓

$= kAe^{kt}$

$\therefore \text{LHS} = \text{RHS}$

$\therefore T = -19 + Ae^{kt}$ satisfies the equation

when $t=0$ $T=18$

$18 = -19 + Ae^{k\times 0}$

$37 = Ae^0$

$A = 37$ ✓

done well

/3 ii) find k: $T=3$ $t=5$

$\therefore 3 = -19 + 37e^{k\times 5}$

$22 = 37e^{k\times 5}$

$\frac{22}{37} = e^{k\times 5}$

$\ln\left(\frac{22}{37}\right) = k\times 5$

$k = \frac{1}{5}\ln\left(\frac{22}{37}\right)$ ✓ $(-0.1039\ldots)$

$\therefore$ find t when $T = -18.9$

$-18.9 = -19 + 37e^{kt}$ ✓

$0.1 = 37e^{k\times t}$

$\frac{0.1}{37} = e^{kt}$

$\ln\left(\frac{0.1}{37}\right) = kt$

$t = \frac{\ln\left(\frac{0.1}{37}\right)}{k}$

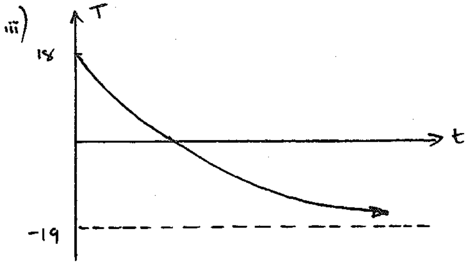
$= 56.87\ldots$ ✓

$\therefore 57$ minutes

Reasoning - 3

done well by nearly all candidates. Errors were made by using incorrect values for T & t

/1 iii)



✓

Communication-1

Q3 a) i) $e^x = x+2 \rightarrow e^x - x - 2 = 0$

/1 Let $f(x) = e^x - x - 2$

$f(1) = e^1 - 1 - 2$ $f(2) = e^2 - 2 - 2$

$= -0.281\ldots < 0$ $= 3.389\ldots > 0$ ✓

since $f(1) < 0$ and $f(2) > 0$ and $f(x)$ is continuous then $f(x)$ has a root between $x=1$ and $x=2$.

Communication - 1

curve must be continuous

/3 ii) $f'(x) = e^x - 1$

$x_2 = 1.5 - \dfrac{f(1.5)}{f'(1.5)}$ ✓

$= 1.5 - \dfrac{e^{1.5} - 1.5 - 2}{e^{1.5} - 1}$ ✓

$= 1.218$ (to 3 dec. pl.) ✓

(correct rounding)

/3 b) $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{(1+x^2)^{3/2}}$ $x = \tan\theta$ $x = \sqrt{3}$ $\theta = \frac{\pi}{3}$

$dx = \sec^2\theta\, d\theta$ $x = \frac{1}{\sqrt{3}}$ $\theta = \frac{\pi}{6}$

$= \displaystyle\int_{\frac{\pi}{6}}^{\pi/3} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}}$ ✓

$= \displaystyle\int_{\frac{\pi}{6}}^{\pi/3} \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{3/2}}$

$= \displaystyle\int_{\frac{\pi}{6}}^{\pi/3} \frac{d\theta}{\sec\theta}$

$= \displaystyle\int_{\frac{\pi}{6}}^{\pi/3} \cos\theta\, d\theta$ ✓

$= \sin\theta \Big]_{\pi/6}^{\pi/3}$

$= \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}$

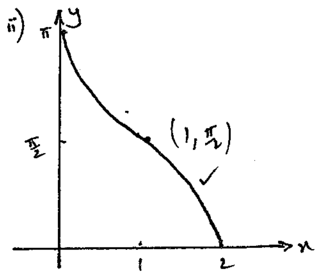
$= \frac{\sqrt{3}-1}{2}$ ✓

Calc - 3

/1 c) i) $-1 \le x-1 \le 1$

$0 \le x \le 2$ ✓

/2 ii)



✓

1 - for graph

1 - for labelling

Communication - 2

/3 iii) $y = \cos^{-1}(x-1)$

$\cos y = x - 1$

$x = \cos y + 1$

$V = \pi \displaystyle\int_0^{\pi} (\cos y + 1)^2\, dy$ ✓

$= \pi \displaystyle\int_0^{\pi} \cos^2 y + 2\cos y + 1\, dy$ now $\cos 2y = 2\cos^2 y - 1$

$\cos^2 y = \frac{1}{2}(1 + \cos 2y)$

$= \pi \displaystyle\int_0^{\pi} \frac{1}{2}\cos 2y + 2\cos y + \frac{3}{2}\, dy$

$= \pi \left[\frac{1}{4}\sin 2y + 2\sin y + \frac{3}{2}y \right]_0^{\pi}$ ✓

$= \pi \left[\left(\frac{1}{4}\sin 2\pi + 2\sin \pi + \frac{3}{2}\pi \right) - (0+0+0) \right]$

$= \pi \times \frac{3}{2}\pi$

$= \frac{3\pi^2}{2}$ units^3 ✓

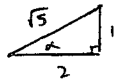
Reasoning - 3

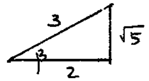
Q4 a) /3

let $\alpha = \tan^{-1}\left(\frac{-1}{2}\right)$ $\qquad \beta = \cos^{-1}\left(\frac{2}{3}\right)$

$\tan\alpha = \frac{-1}{2}$ ✓ $\qquad \cos\beta = \frac{2}{3}$

$\therefore \alpha$ in 4th quad.





$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ ✓

$= \frac{-1}{\sqrt5} \times \frac{2}{3} + \frac{2}{\sqrt5} \times \frac{\sqrt5}{3}$

$= \frac{-2}{3\sqrt5} + \frac{2}{3}$

$= \frac{-2\sqrt5}{15} + \frac{2}{3}$

$= \frac{10 - 2\sqrt5}{15}$ ✓

Reasoning - 3

Done well and most students recognised α was in 4th quad.

/1 b) i)

RHS $= 1 - \frac{1}{u+1}$

$= \frac{u+1-1}{u+1}$ ✓

$= \frac{u}{u+1}$

/2 ii) $\int \frac{dx}{1+\sqrt{x}}$ $\qquad x = u^2$, $dx = 2u\,du$

$= \int \frac{2u\,du}{1+u}$ ✓

$= 2\int \frac{u\,du}{1+u}$

$= 2\int 1 - \frac{1}{u+1}\,du$

$= 2\left(u - \ln|u+1|\right) + c$

$= 2\sqrt{x} - 2\ln(1+\sqrt{x}) + c$ ✓

Calc - 2

You must remember to replace u with $\sqrt{x}$ after integrating.

/1 c) i) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}(e^x - e^{-x})\right)$

$= \frac{1}{2}(e^x + e^{-x})$ ✓

Done well

/1 ii) since $e^x > 0$ and $e^{-x} > 0$ for all x

$\frac{dy}{dx} > 0$ for all x

$\therefore y = \sinh x$ is a monotonic increasing function

$\therefore y = \sinh x$ has an inverse fn. ✓

Communication - 1

Very poor reasoning here. You must link $\frac{dy}{dx} > 0$ with monotonic increasing

/4 iii) $y = \frac{1}{2}(e^x - e^{-x})$

interchange x and y

$x = \frac{1}{2}(e^y - e^{-y})$ ✓

$2x = e^y - \frac{1}{e^y}$

$2xe^y = e^{2y} - 1$

$e^{2y} - 2xe^y - 1 = 0$ ✓

let $m = e^y$

$m^2 - 2xm - 1 = 0$

$m = \frac{2x \pm \sqrt{(2x)^2 - 4.1.-1}}{2}$

$= \frac{2x \pm \sqrt{4x^2+4}}{2}$

$= \frac{2x \pm 2\sqrt{x^2+1}}{2}$

$= x \pm \sqrt{x^2+1}$ ✓

$\therefore e^y = x \pm \sqrt{x^2+1}$

since $\sqrt{x^2+1} > x$ ✓

$e^y = x + \sqrt{x^2+1}$

$y = \ln\left(x + \sqrt{x^2+1}\right)$

Reasoning - 4

Very few candidates knew how to progress past $x = \frac{1}{2}(e^y - e^{-y})$