SCEGGS Darlinghurst

2010

HSC Assessment 2
11th June, 2010

Mathematics Extension 1

Outcomes Assessed: PE2, PE3, HE4, HE6 and HE7

General Instructions

- Time allowed – 70 minutes
- This paper has **four** questions
- Attempt **all** questions
- Answer all questions on the pad paper provided
- Begin each question on a **new page**
- Write your Student Number at the top of each page
- Mathematical templates, geometrical equipment and approved scientific calculators may be used
- A table of standard integrals is provided

Question	Calculus	Communication	Reasoning	Marks
1	/2	/3		/12
2	/2	/3	/2	/12
3	/5		/6	/12
4	/5	/1	/4	/12
TOTAL	**/15**	**/7**	**/12**	**/48**

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Total marks – 48
Attempt Questions 1–4

Answer each question on the pad paper provided.
Write your student number at the top of each page.
Begin each question on a NEW page.

Marks

Question 1 (12 marks)

(a) If α, β, and γ are the roots of $P(x) = 2x^3 - x^2 - 8x + 4$, find **5**

(i) $\alpha + \beta + \gamma$

(ii) $\alpha\beta + \beta\gamma + \alpha\gamma$

(iii) $\alpha\beta\gamma$

(iv) $\alpha^2 + \beta^2 + \gamma^2$

(b) (i) State the domain and range of the function $f(x) = 3\cos^{-1} 2x$. **2**

(ii) Draw a neat sketch of the function $f(x) = 3\cos^{-1} 2x$, clearly labeling important features. **1**

(c) Find the exact value of $\displaystyle\int_0^4 \frac{3}{\sqrt{16 - x^2}}\, dx$. **2**

(d) A function is given by the rule $f(x) = \dfrac{x+1}{x+2}$. **2**

Find the rule for the inverse function $f^{-1}(x)$.

End of Question 1

Marks

Question 2 (12 marks) Begin a NEW page.

(a) (i) Sketch $y = 3\sin x$ and $y = x$ for $0 \le x \le 2\pi$ on the same set of axis. **1**

(ii) By considering $f(x) = 3\sin x - x$, show that the curve $y = 3\sin x$ and the line $y = x$ meet at a point P whose x coordinate is between $x = 2.2$ and $x = 2.4$. **1**

(iii) Using one application of Newton's method, starting at $x = 2.3$, find an approximation for the x coordinate of P. Give your answer to two decimal places. **2**

(b) (i) Find the exact value of $\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(-1)$. **1**

(ii) Hence or otherwise, find the area bounded by the curve $y = \dfrac{1}{4 + x^2}$, the x-axis and the ordinates $x = -2$ and $x = 2\sqrt{3}$. **2**

(c) Find the exact value of $\cos\left(\sin^{-1}\left(-\dfrac{3}{4}\right)\right)$. **2**

(d) (i) Given $P(x) = x^3 + 3x^2 - 10x - 24$, show that $(x + 2)$ is a factor of $P(x)$ and express $P(x)$ as the product of its linear factors. **2**

(ii) Hence solve the inequality $x^3 + 3x^2 - 10x > 24$ **1**

End of Question 2

Marks

Question 3 (12 marks) Begin a NEW page.

(a) Use the substitution $u = 1 - x$ to find the exact value of $\int_0^1 x\sqrt{1-x}\ dx$. **3**

(b) At any point on the curve $y = f(x)$ the gradient function is given by $\frac{dy}{dx} = 2\cos^2 x + 1$. If $y = \pi$ when $x = \pi$, find the value of y when $x = 2\pi$. **4**

(c) (i) Use long division to show that **1**

$$\frac{x^3 - 2x^2 + 3x - 1}{x^2 + 3} = x - 2 + \frac{5}{x^2 + 3}$$

(ii) Hence find an expression for **2**

$$\int \frac{x^3 - 2x^2 + 3x - 1}{x^2 + 3}\ dx$$

(d) Use an appropriate compound angle formula to find the exact value of **2**

$$\cos\left(\tan^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{4}\right)$$

End of Question 3

Marks

Question 4 (13 marks) Begin a NEW page.

(a) (i) Find $\dfrac{d}{dx}\left(x\tan^{-1}x\right)$. **1**

(ii) Hence or otherwise find the exact value of $\displaystyle\int_0^1 \tan^{-1}x \ \ dx$. **3**

(b) Use the substitution $u = e^x$ or otherwise show that **3**

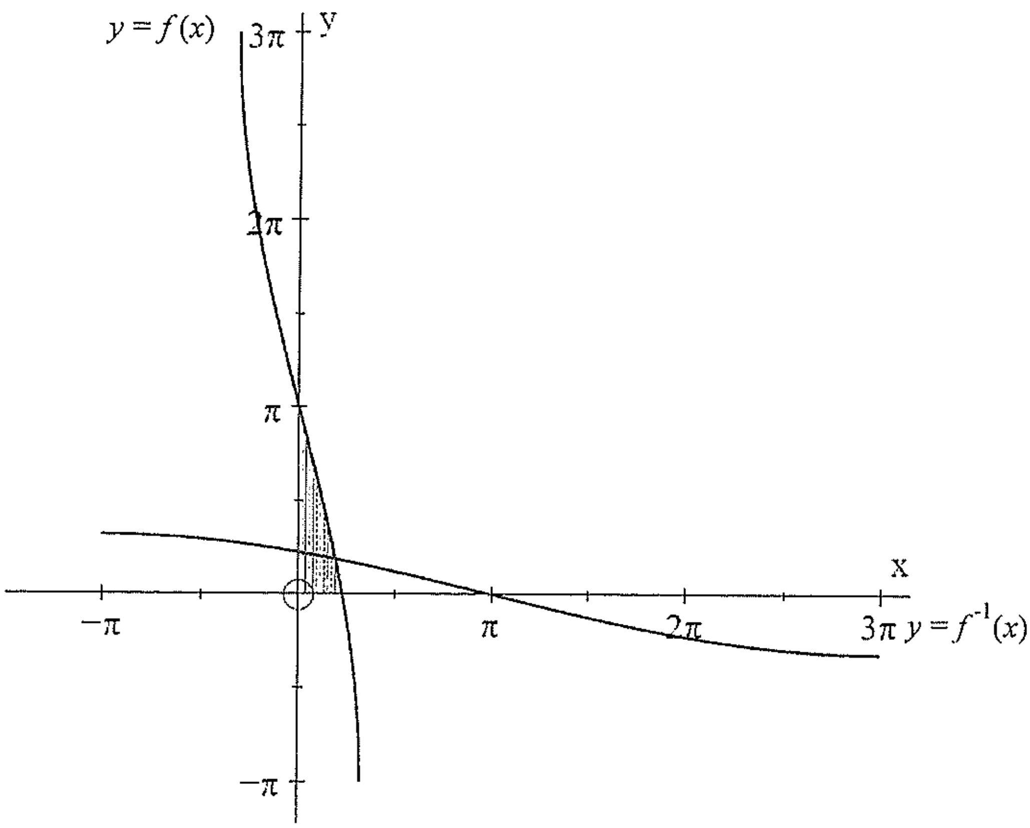
$$\int_0^{\ln 10} \frac{3}{1+2e^{-x}} \ dx \ = 6\ln 2$$

Question 4 continues on the next page

Question 4 continued.

Marks

(c)



The graph shows the curves $y = f(x)$ and its inverse $y = f^{-1}(x)$ where $f(x) = \pi - 4\sin^{-1}x$

(i) Find the exact value of x where the curve $y = f(x)$ cuts the x axis. **1**

(ii) Find the equation of the inverse function $y = f^{-1}(x)$. **1**

(iii) Explain why the area bounded by $y = f(x)$ in the first quadrant is given by **1**

$$Area = \int_0^{\pi} \sin\left(\frac{\pi}{4} - \frac{x}{4}\right) dx$$

(iv) Find the exact value of the area. **2**

End of Paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

HSC - Extension 1 Task 2 2010 - Solutions

Question 1 (12 marks)

calc 2 com 3

part(iv) caused some problems. Ensure you know this rule or know how to get to it

a) $P(x) = 2x^3 - x^2 - 8x + 4$

i) $\alpha + \beta + \gamma = \frac{1}{2}$ ✓

ii) $\alpha\beta + \beta\gamma + \alpha\gamma = -4$ ✓

iii) $\alpha\beta\gamma = -2$ ✓

iv) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$ ✓

$= \left(\frac{1}{2}\right)^2 - 2(-4)$

$= 8\frac{1}{4}$ ✓

b) $f(x) = 3\cos^{-1} 2x$

i) $D: -1 \leq 2x \leq 1$

$-\frac{1}{2} \leq x \leq \frac{1}{2}$ ✓

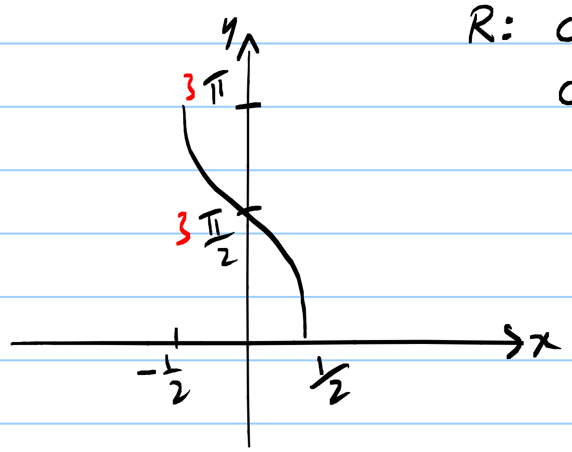
$R: 0 \leq \frac{y}{3} \leq \pi$

$0 \leq y \leq 3\pi$ ✓

com 3

Very well done!

ii)



✓

c) $\displaystyle\int_0^4 \frac{3}{\sqrt{16 - x^2}}\,dx = 3\int_0^4 \frac{dx}{\sqrt{4^2 - x^2}}$

$= 3\left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$ ✓

Ca 2

$= \frac{3\pi}{2}$ ✓

d) $f(x) = \frac{x+1}{x+2}$

$f^{-1}(x)$: $x = \frac{y+1}{y+2}$

$xy + 2x = y+1$ ✓

$xy - y = 1-2x$

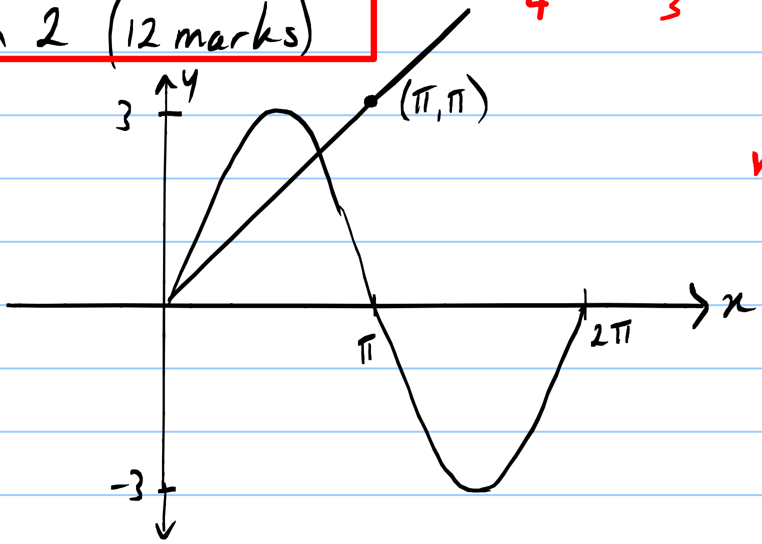
$y(x-1) = 1-2x$

$y = \frac{1-2x}{x-1}$ ✓

Calc 4 Com 3 reas 2

Question 2 (12 marks)

a) i)



✓ Comm 1

The location of the line $y=x$ was not well done. Find some points on your calculator $(\frac{\pi}{2},\frac{\pi}{2})$ (π,π) and plot the location carefully

ii) $f(x) = 3\sin x - x$

$f(2.2) = 0.225$
$f(2.4) = -0.374$

to find the point of intersection we solve $y = 3\sin x$ & $y = x$ simultaneously, this results in the equation $3\sin x - x = 0$
since $f(2.2) > 0$ and $f(2.4) < 0$ and the curve is continuous, there must be a solution between these values.

Comm 1

Some reasons were too brief. Here are all the details you should include.

iii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 2.3 - \frac{(3\sin 2.3 - 2.3)}{(3\cos 2.3 - 1)}$ ✓

$= 2.28$ ✓

Calc 2

- Don't round off too early.
- Make sure your calculator is in radians.

Write the formula so you don't get mixed up.

b) i) $\tan^{-1}(\sqrt{3}) - \tan^{-1}(-1)$

$= \frac{\pi}{3} + \frac{\pi}{4}$

$= \frac{7\pi}{12}$

Just use your calculator because they are exact ratios. It's too messy using tan(A-B)

ii) $A = \int_{-2}^{2\sqrt{3}} \frac{1}{4+x^2}\, dx$

$= \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2\sqrt{3}}$

$= \frac{1}{2}\left(\tan^{-1}\sqrt{3} - \tan^{-1}(-1)\right)$

$= \frac{1}{2} \times \frac{7\pi}{12}$

$= \frac{7\pi}{24}$

Calc 2

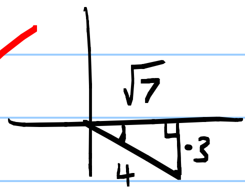
This is a very basic integration directly from the table of standard integrals.

No one should get this wrong.

c) $\cos\left(\sin^{-1}\left(-\frac{3}{4}\right)\right)$

let $x = \sin^{-1}\left(-\frac{3}{4}\right)$ Quadrant 4

$\sin x = -\frac{3}{4}$



$\therefore \cos x = \frac{\sqrt{7}}{4}$

Reas 2

It is highly recommended to draw the diagram in the correct location clearly showing signs in Quadrant 4.

$\Rightarrow$ greater success

(d) $P(x) = x^3 + 3x^2 - 10x - 24$

$P(-2) = -8 + 12 + 20 - 24$

$= 0$

$\therefore (x+2)$ is a factor ✓

$\boxed{\text{* make a conclusion}}$

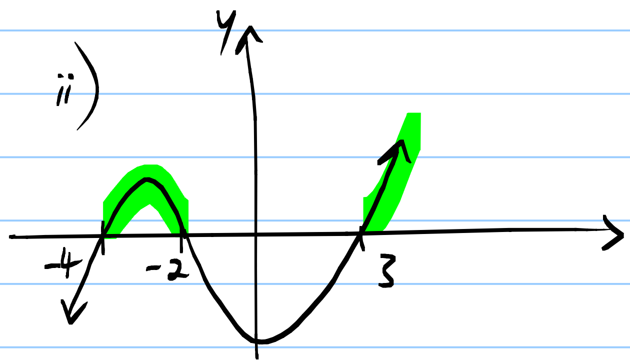
$$\begin{array}{r} x^2 + x - 12 \\ x+2 \overline{) x^3 + 3x^2 - 10x - 24} \\ \underline{x^3 + 2x^2} \\ x^2 - 10x \\ \underline{x^2 + 2x} \\ -12x - 24 \\ \underline{-12x - 24} \\ 0 \end{array}$$

$\therefore P(x) = (x+2)(x^2+x-12)$

$= (x+2)(x-3)(x+4)$ ✓

It was most disappointing to see any students unable to do long division. You must practise this technique. It will very likely appear in both the Trial HSC and HSC.

ii)



$x^3 + 3x^2 - 10x > 24$

$x^3 + 3x^2 - 10x - 24 > 0$

$-4 < x < 2, \quad x > 3$ ✓ com

1

This is a very standard basic question. If you couldn't do it please do more practice on Polynomials before the Trial & HSC.

Question 3 (12 marks)

Calc 5 Reas 6

a) $u = 1-x$, $x = 1-u$

$\frac{du}{dx} = -1$

$du = -dx$

$x=0, u=1$

$x=1, u=0$

Calc 3

$$\int_0^1 x\sqrt{1-x}\,dx$$

$$= -\int_1^0 (1-u)u^{\frac{1}{2}}\,du$$

$$= -\int_1^0 u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$$

$$= \int_1^0 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$$

$$= \left[\frac{2}{5}u^{5/2} - \frac{2u^{3/2}}{3}\right]_1^0$$

$$= 0 - \left(\frac{2}{5} - \frac{2}{3}\right)$$

$$= \frac{4}{15}$$

You can also use the fact that $\int_a^b = -\int_b^a$ to change this here $-\int_1^0 = \int_0^1$ to make it easier.

b) $\frac{dy}{dx} = 2\cos^2 x + 1$

$$y = \int 2\cos^2 x + 1\,dx$$

$$= \int 2\left(\tfrac{1}{2}(1+\cos 2x)\right) + 1\,dx$$

$$= \int (1 + \cos 2x + 1)\,dx$$

$$= \int (\cos 2x + 2)\,dx$$

* $\cos 2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta = \cos 2\theta + 1$

* You <u>must</u> know the substitutions

$\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)\,dx$

$\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx$

<u>and</u> how to apply them

$$y = \frac{1}{2}\sin 2x + 2x + c$$

$$\pi = \frac{1}{2}\sin 2\pi + 2\pi + c$$
$$\pi = 2\pi + c$$
$$c = -\pi$$

$$\therefore \quad y = \frac{1}{2}\sin 2x + 2x - \pi$$

at $x = 2\pi$

$$y = \frac{1}{2}\sin 4\pi + 4\pi - \pi$$
$$y = 3\pi$$

Evaluate $\sin 2\pi = 0$ to simplify at this stage. Quite a few students made it harder by not doing that step.

Reas 4

c) i)

$$\begin{array}{r|l} & x \quad -2 \\ \hline x^2+3 & x^3 - 2x^2 + 3x - 1 \\ & x^3 \qquad\quad + 3x \\ \hline & -2x^2 \qquad\quad - 1 \\ & -2x^2 \qquad\quad - 6 \\ \hline & 5 \end{array}$$

Long division again! You **must** be able to do it

$$\therefore \quad x^3 - 2x^2 + 3x - 1 = (x^2+3)(x-2) + 5$$
$$\frac{x^3 - 2x^2 + 3x - 1}{x^2+3} = x - 2 + \frac{5}{x^2+3}$$

ii)

$$\int \frac{x^3 - 2x^2 + 3x - 1}{x^2+3}\, dx$$

$$= \int x - 2 \, dx + \int \frac{5}{x^2+3}\, dx$$

$$= \frac{x^2}{2} - 2x + 5\int \frac{1}{x^2+(\sqrt{3})^2}\, dx$$

$$= \frac{x^2}{2} - 2x + \frac{5}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$$

* Everyone should have recognised this integration using the table of standard integrals especially in an Inverse Functions test.

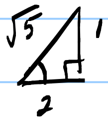
Calc 2

d) $\cos\left(\tan^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{4}\right)\right)$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

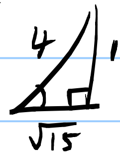
let $\alpha = \tan^{-1}\frac{1}{2}$ $\qquad$ $\beta = \sin^{-1}\frac{1}{4}$

$\tan\alpha = \frac{1}{2}$ $\qquad$ $\sin\beta = \frac{1}{4}$



$\cos\alpha = \frac{2}{\sqrt{5}}$

$\sin\alpha = \frac{1}{\sqrt{5}}$



$\cos\beta = \frac{\sqrt{15}}{4}$ ✓

$\therefore \cos(\alpha+\beta) = \frac{2}{\sqrt{5}} \times \frac{\sqrt{15}}{4} - \frac{1}{\sqrt{5}} \times \frac{1}{4}$

$= \frac{\sqrt{3}}{2} - \frac{1}{4\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{\sqrt{3}}{2} - \frac{\sqrt{5}}{20}$

$= \frac{10\sqrt{3} - \sqrt{5}}{20}$ ✓

Reas 2

This part was pretty well done. Make sure you can rationalise surds.

Question 4 (13 marks)

Calc 5 Com 1 R 4

a) i) $\frac{d}{dx} x\tan^{-1}x$

$u = x \quad v = \tan^{-1}x$

$u' = 1 \quad v' = \frac{1}{1+x^2}$

$= \tan^{-1}x + \frac{x}{1+x^2}$ ✓

Well done!

ii) Hence $\int_0^1 \tan^{-1}x + \frac{x}{1+x^2}\, dx = x\tan^{-1}x$

R 4

$\int_0^1 \tan^{-1}x\, dx + \int_0^1 \frac{x}{1+x^2}\, dx = \left[x\tan^{-1}x\right]_0^1$ ✓

$\int_0^1 \tan^{-1}x\, dx = \left[x\tan^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\, dx$

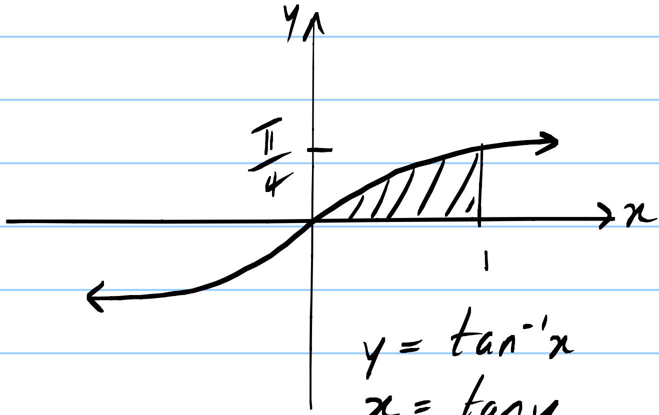
$= \left[x\tan^{-1}x\right]_0^1 - \frac{1}{2}\left[\log_e(1+x^2)\right]_0^1$ ✓

$= \left(\frac{\pi}{4}\right) - \frac{1}{2}\left(\log_e 2 - \log_e 1\right)$

$= \frac{\pi}{4} - \log_e \sqrt{2}$ ✓

many students were able to determine the initial line of working but then had difficulty determining which part to integrate and with the integration itself. You should recognise that $\int \frac{x}{1+x^2}$ will be a log

alternate method



$$\int_0^1 \tan^{-1}x = \text{rectangle} - \int_0^{\frac{\pi}{4}} \tan y \, dy$$

$$= \frac{\pi}{4} + \int_0^{\frac{\pi}{4}} \frac{-\sin y}{\cos y} \, dy$$

$$= \frac{\pi}{4} + \left[\log_e(\cos y)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} + \left(\log_e \frac{1}{\sqrt{2}} - \log_e 1\right)$$

$$= \frac{\pi}{4} + \log_e \sqrt{2}^{-1}$$

$$= \frac{\pi}{4} - \log_e \sqrt{2}$$

6) $u = e^x$ $\qquad \frac{1}{u} = \frac{1}{e^x}$

$\frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

when $x = 0, u = 1$

$x = \ln 10, u = 10$

$$\int_0^{\ln 10} \frac{3}{1 + 2e^{-x}} \, dx = \int_1^{10} \frac{3}{1 + \frac{2}{u}} \times \frac{du}{u}$$

$$= \int_1^{10} \frac{3}{\frac{u+2}{u}} \times \frac{du}{u}$$

$$= 3\int_1^{10} \frac{\cancel{u}}{u+2} \times \frac{du}{\cancel{u}}$$

$$= 3\int_1^{10} \frac{1}{u+2} \, du$$

$$= 3\left[\log_e(u+2)\right]_1^{10}$$

$$= 3(\log_e 12 - \log_e 3)$$

$$= 3\log_e 4$$

$$= 3\log_e 2^2$$

$$= 6\ln 2$$

> The substitution here was a little tricky because e^{-x} was in the denominator and the fractions caused problems for some students. Most know what needs to be done but are getting caught with the algebra

Ca
3

alternate method

$$\int_0^{\ln 10} \frac{3}{1+\frac{2}{e^x}}\,dx$$

$$= \int_0^{\ln 10} \frac{3}{\frac{e^x+2}{e^x}}\,dx \checkmark$$

$$= 3\int_0^{\ln 10} \frac{e^x}{e^x+2}\,dx$$

$$= 3\left[\log_e\left(e^x+2\right)\right]_0^{\ln 10} \checkmark$$

$$= 3\left(\log_e\left(e^{\ln 10}+2\right)-\log_e 3\right)$$

$$= 3\left(\log_e 12 - \log_e 3\right)$$

$$= 3\log_e 4 = 6\ln 2 \checkmark$$

c) i) $f(x) = \pi - 4\sin^{-1}x$

$$0 = \pi - 4\sin^{-1}x$$

$$4\sin^{-1}x = \pi$$

$$\sin^{-1}x = \frac{\pi}{4}$$

$$x = \sin\frac{\pi}{4}$$

$$x = \frac{1}{\sqrt{2}} \checkmark$$

parts (i) & (ii) were very well done. Explanations in part (iii) were often too brief you needed to explain 2 things, first why 0 to π and secondly why $\sin\left(\frac{\pi}{4}-\frac{x}{4}\right)$ gave you the required area

ii) inverse $x = \pi - 4\sin^{-1} y$

$$\sin^{-1} y = \frac{\pi}{4} - \frac{x}{4}$$

$$y = \sin\left(\frac{\pi}{4} - \frac{x}{4}\right)$$

iii) The area bounded by $y = f(x)$ in the first quadrant would be given by $\int_0^{\frac{\pi}{4}} \pi - 4\sin^{-1} x \, dx$

but this is not easily calculated.

the area bounded by $f(x)$ in the first quadrant is equal to the area bounded by $f(x)$ and the y axis from $y=0$ to $y=\pi$

com 1

This area is also equal to the area bounded by $f^{-1}(x)$ and the x-axis from $x=0$ to $x=\pi$

ie $\text{Area} = \int_0^{\pi} \sin\left(\frac{\pi}{4} - \frac{x}{4}\right) dx$

iv) $\text{Area} = \int_0^{\pi} \sin\left(\frac{\pi}{4} - \frac{x}{4}\right) dx$

$$= \left[+4\cos\left(\frac{\pi}{4} - \frac{x}{4}\right)\right]_0^{\pi}$$

Ca 2

$$= 4\left(\cos 0 - \cos\frac{\pi}{4}\right)$$

$$= 4\left(1 - \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$= 4\left(\frac{2-\sqrt{2}}{2}\right)$$

$$= 4 - 2\sqrt{2} \quad u^2$$

quite good but some careless errors with negative signs