Centre Number

Student Number

SCEGGS Darlinghurst

2016

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 2

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 1½ hours
- Write using black pen
- Write your Student Number on the front of each writing booklet
- Attempt **all** questions and show all necessary working
- Marks may be deducted for careless or badly arranged work
- Mathematical templates and geometrical equipment and scientific calculators may be used

Total marks – 52

Section I

7 marks

- Attempt Questions 1–7
- Allow about 18 minutes for this section
- Answer on the separate Multiple Choice Answer Sheet provided

Section II

45 marks

- Attempt Questions 8–10
- Allow about 72 minutes for this section
- ALL working must be shown
- Answer in the writing booklets provided
- Start each question in a **new booklet**

Question/s	Binomial	Further Integration	TOTAL
1–7 Objective Response	/1		/7
8		/4	/15
9	/3	/3	/15
10	/4	/3	/15
TOTAL	**/8**	**/10**	/52

Section I – Multiple Choice

7 marks
Attempt Questions 1–7
Allow about 18 minutes for this section

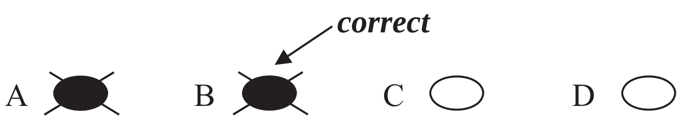
Use the multiple-choice answer sheet provided.

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ***correct*** and drawing an arrow as follows.



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1 If A and B are the points $(2, -1)$ and $(-2, -4)$ respectively, what are the co-ordinates of the point P which divides the interval AB externally in the ratio $1:3$?

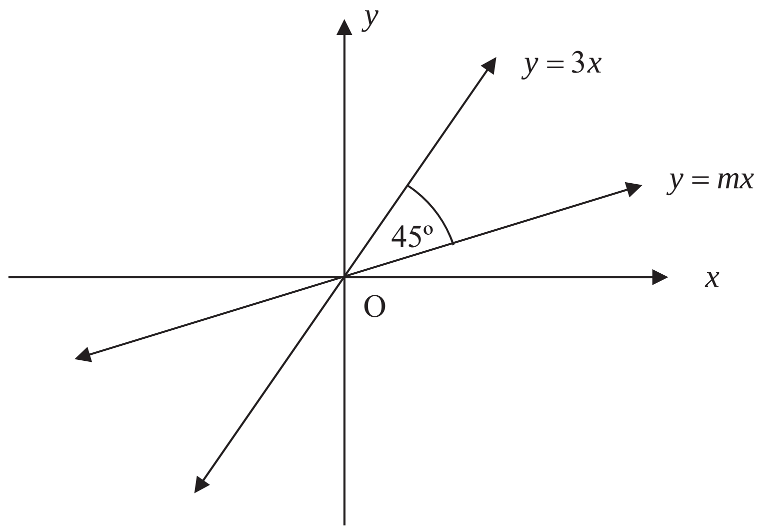
(A) $\left(4, \frac{1}{2}\right)$

(B) $\left(-2, \frac{-11}{2}\right)$

(C) $\left(-2, \frac{-7}{4}\right)$

(D) $\left(2, \frac{-13}{4}\right)$

2



The angle between $y = 3x$ and $y = mx$ is 45º. What is the value of m?

(A) $\frac{1}{3}$

(B) $\frac{1}{2}$

(C) 1

(D) $1\frac{1}{2}$

3 The equation of the tangent to the curve $ay = x^2$ at the point where $x = 2a$ is

(A) $4x - y - 4a = 0$

(B) $4x - y = 0$

(C) $4ax - y - 4a^2 = 0$

(D) $x + 4y - 18a = 0$

4 The equation of the chord of contact of the parabola $y = x^2$ from the point $(1, -1)$ is

(A) $x + 8y + 8 = 0$

(B) $x - 8y + 8 = 0$

(C) $2x + y + 1 = 0$

(D) $2x - y + 1 = 0$

5 When the polynomial $P(x)$ is divided by $(x+3)(x-4)$ the remainder is $3x + 2$. What is the remainder when $P(x)$ is divided by $x - 4$?

(A) –10

(B) –7

(C) 11

(D) 14

6 Which of the following is the co-efficient of x^n in the expansion of

$$(1+x)^n + (1-x)^n ?$$

(A) 0

(B) $\dfrac{1+(-1)^n}{2}$

(C) $1+(-1)^n$

(D) 2

7 A polynomial $P(x) = x^3 + 4x^2 - x + 5$ has zeros α, β and γ.

A second polynomial $Q(x) = x^3 + bx^2 + cx + d$ has zeros $3\alpha, 3\beta$ and 3γ.

What are the values of b and d?

(A) $b = 12, d = 135$

(B) $b = -12, d = -135$

(C) $b = 12, d = 15$

(D) $b = -12, d = -15$

End of Section I

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Section II

45 marks
Attempt Questions 8–10
Allow about 72 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 8 (15 marks)

(a) Find $\int \cos^2 \frac{x}{2}\, dx$ **2**

(b) Solve $\frac{3x}{x-2} \geq 4$ **3**

(c) If 5 people are to be chosen from a group of 6 men and 4 women, find the probability that at least 2 women are chosen. **3**

(d) Find $\int \frac{5x}{\left(3-5x^2\right)^3}\, dx$ using the substitution $u = 3 - 5x^2$. **2**

(e) (i) Prove **2**

$$\sin 2x - \tan x \cos 2x = \tan x$$

(ii) Hence prove $\tan \frac{3\pi}{8} = \sqrt{2} + 1$. **3**

■ Start a new writing booklet

Question 9 (15 marks)

(a) (i) Prove that the equation $x^5 + 2x - 20 = 0$ has only one real root. **1**

(ii) Confirm that the real root lies between $x = 1$ and $x = 2$. **1**

(iii) Using $x_0 = 1.5$ as a first approximation take one application of Newton's Method to obtain a closer approximation correct to 2 decimal places. **2**

(b) Find the greatest co-efficient in the expansion of **3**

$$\left(2x^2 + \frac{3}{x}\right)^{12}$$

Answer in index form.

(c) Find the volume formed when the curve $y = \sin 2x$ from $x = 0$ to $x = \pi$ is rotated about the x axis. **3**

(d) Consider the curve $y = \dfrac{x^2}{1 - x^2}$.

(i) Find any horizontal or vertical asymptotes. **2**

(ii) Find any stationary points. **1**

(iii) Sketch the curve showing important features. **2**

■ Start a new writing booklet

Question 10 (15 marks)

(a) Evaluate **3**

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta \, d\theta}{\sqrt{4 + 4\tan\theta}}$$

using the substitution $u = 1 + \tan\theta$.

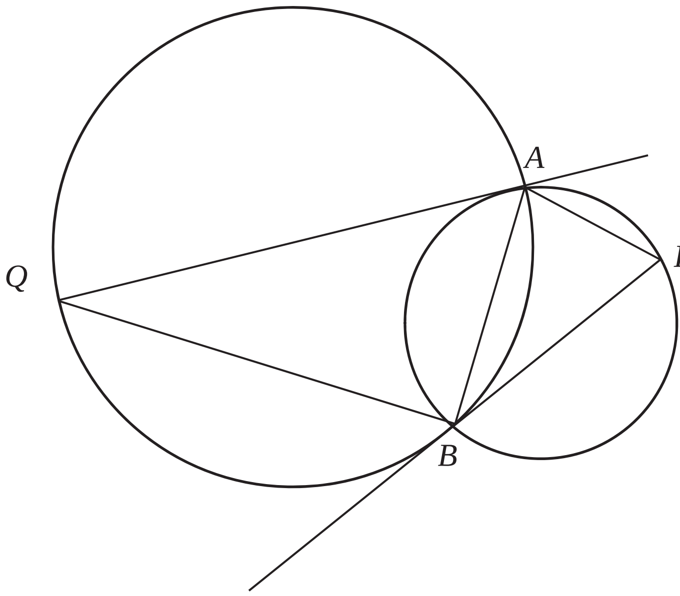
(b) (i) Prove by mathematical induction **3**

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n}{6}(n+1)(2n+1) \text{ for } n \text{ integer } n \geq 1$$

(ii) Hence evaluate **1**

$$\lim_{n \to \infty} \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}$$

(c)



Two circles intersect at A and B, forming a common chord, AB. Tangents are drawn through A and B as shown opposite, meeting the circles in Q and P respectively.

Not to scale

(i) Prove that the triangles AQB and PAB are similar. **2**

(ii) Hence, or otherwise, prove $AB^2 = PA \times BQ$. **1**

(iii) What relationship exists between AP and QB? Justify your answer. **1**

Question 10 continues on the next page

Question 10 (continued)

(d) A bag contains n apples p of which are ripe.

(i) If two apples are withdrawn from the bag, the first being replaced before the second is withdrawn, prove that the probability that at least one is ripe is **2**

$$\frac{p\left(2n-p\right)}{n^2}$$

(ii) If eight apples are withdrawn, each being replaced, as before, show that the probability that at least two are ripe is **2**

$$1-\left(n-p\right)^7\frac{\left(n+7p\right)}{n^8}$$

End of paper

2016 HSC COURSE ASSESSMENT TASK 2

Mathematics Extension 1

Centre Number

Student Number

Section I

7 marks
Attempt Question 1–7
Allow about 18 minutes for this section

Write your Student Number at the top of this page.

Multiple Choice Answer Sheet

Question					
	1	A ○	B ○	C ○	D ○
	2	A ○	B ○	C ○	D ○
	3	A ○	B ○	C ○	D ○
	4	A ○	B ○	C ○	D ○
	5	A ○	B ○	C ○	D ○
	6	A ○	B ○	C ○	D ○
	7	A ○	B ○	C ○	D ○

Solutions 2016 Extension 1 Assessment 2.

1. $(2,-1)$ $(-2,4)$ $-1:3$

$x = \dfrac{2\times 3 - 1\times -2}{2}$ $\qquad y = \dfrac{-1\times 3 + 4\times -1}{2}$ **A**

$= 4$ $\qquad = \dfrac{1}{2}$

2. $m_1 = 3$ $m_2 = m$ $\tan 45° = \left|\dfrac{3-m}{1+3m}\right| = 1$

$3 - m = 1 + 3m$ or $3 - m = -1 - 3m$

$-4m = -2$ $\qquad 2m = -4$ **B**

$m = \dfrac{1}{2}$ $\qquad m = -2$

3. $y = \dfrac{x^2}{a}$ $\qquad y = \dfrac{4a^2}{a} = 4a$

$y' = \dfrac{2x}{a}$ at $x = 2a$, $y' = \dfrac{4a}{a} = 4$

equation: $y - 4a = 4(x - 2a)$

$= 4x - 8a$. **A.**

$4x - y - 4a = 0$

4. $a = \dfrac{1}{4}$ chord: $xx_1 = 2a(y + y_1)$

$x = \dfrac{1}{2}(y - 1)$

D.

$2x - y + 1 = 0.$

5. $P(x) = (x+3)(x-4)Q(x) + 3x + 2$

$P(4) = 0 + 4\times 3 + 2 = 14.$ **D.**

6. $x^n + (-1)^n x^n$ coefficient $1 + (-1)^n$ BT **c**

7. $\alpha + \beta + \gamma = -4$, $\alpha\beta + \alpha\gamma + \beta\gamma = -1$ $\alpha\beta\gamma = -5.$

zeros $3\alpha, 3\beta, 3\gamma$. $\qquad 9\alpha\beta + 9\alpha\gamma + 9\beta\gamma = -9$

$3\alpha + 3\beta + 3\gamma = -12$ $\qquad 27\alpha\beta\gamma = -135.$

$x^3 + 12x^2 - 9x + 135 = 0$ $\qquad b = 12,\ d = 135$ **A**

$x^3 + bx^2 + cx + d = 0.$

Question 8.

Further Int /2

a) $\int \cos^2 \frac{x}{2}\, dx = \int \frac{1}{2}(1+\cos x)\, dx$ ✓

$= \frac{1}{2}(x + \sin x) + c$ ✓

A few people did not know the formulas.

b) $\frac{3x}{x-2} \geq 4 \qquad (x \neq 2)$

$3x(x-2) \geq 4(x-2)^2$

$\geq 4(x^2-4x+4)$

$3x^2-6x \geq 4x^2-16x+16$

$x^2-10x+16 \leq 0$

$(x-8)(x-2) \leq 0$

$2 < x \leq 8$ ✓✓✓

Generally well done.. a few forgot and wrote $2 \leq x \leq 8$ instead of $2 < x \leq 8$

c) Total number of groups

$= \binom{10}{5} = 252.$ ✓

Well done

number with no women

$= \binom{6}{5} = 6$ ✓

number with one woman

$= \binom{6}{4} \times \binom{4}{1} = 60$

$\therefore$ Probability $= 1 - \frac{66}{252} = \frac{31}{42}$ ✓

d) $u = 3-5x^2$

Further Int /2

$du = -10x\, dx$

$-\frac{1}{2} du = 5x\, dx$

Very well done.

$\int \frac{5x\, dx}{(3-5x^2)^3} = -\frac{1}{2}\int \frac{du}{u^3}$ ✓

$= -\frac{1}{2} \frac{u^{-2}}{-2} + c$

$= \frac{1}{\quad} + c$ ✓

e) $\sin 2x - \tan x \cos 2x$

$= 2\sin x\cos x - \frac{\sin x}{\cos x}(2\cos^2 x - 1)$

$= 2\sin x\cos x - 2\frac{\sin x}{\cos x}\cos^2 x + \frac{\sin x}{\cos x}$ ✓

$= 2\sin x\cos x - 2\sin x\cos x + \tan x$

$= \tan x.$ ✓

Some students used the wrong formula, ie. $1 - 2\sin^2 x$!!

You must look ahead !!

(ii) if $x = \frac{3\pi}{8}$,

$\tan\frac{3\pi}{8} = \sin 2\times\frac{3\pi}{8} - \tan\frac{3\pi}{8}\cos 2\times\frac{3\pi}{8}$

$= \sin\frac{3\pi}{4} - \tan\frac{3\pi}{8}\cos\frac{3\pi}{4}$

$= \frac{1}{\sqrt{2}} - \tan\frac{3\pi}{8}\times\frac{-1}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tan\frac{3\pi}{8}.$

Most got to here !!

$\therefore \tan\frac{3\pi}{8} - \frac{1}{\sqrt{2}}\tan\frac{3\pi}{8} = \frac{1}{\sqrt{2}}.$

$\sqrt{2}\tan\frac{3\pi}{8} - \tan\frac{3\pi}{8} = 1$

$(\sqrt{2}-1)\tan\frac{3\pi}{8} = 1$

$\tan\frac{3\pi}{8} = \frac{1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}$

$= \frac{\sqrt{2}+1}{2-1}$

$= \sqrt{2}+1$

But failed to see...

$\tan\frac{3\pi}{8} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tan\frac{3\pi}{8}$

then to gather like terms and factorize.

Question 9.

a) (i) Let $P(x) = x^5 + 2x - 20$

$P'(x) = 5x^4 + 2$

$5x^4 + 2 > 0$ for all real x

$\therefore$ $P(x)$ is an increasing polynomial with no stationary point

$\therefore$ $P(x) = 0$ has only one real root. as $y = P(x)$ crosses the x axis only once.

The $\Delta (b^2 - 4ac)$ for a quadratic is irrelevant to a quintic polynomial

✓ only a handful of students provided sufficient explanation for this

(ii) $P(1) = 1 + 2 - 20 = -17 < 0$

$P(2) = 32 + 4 - 20 = 16 > 0$

Since the sign has changed and $y = P(x)$ is continuous, there is a root between $x = 1$ and $x = 2$.

Only a few students commented on the continuity ✓

(iii) $P(1.5) = 1.5^5 + 2 \times 1.5 - 20$

$= -9.40625$

$P'(1.5) = 5 \times 1.5^4 + 2$

$= 27.3125$ ✓

$$x_1 = 1.5 - \frac{-9.40625}{27.3125}$$

$= \underline{1.8443 \ldots}$

$x_1 \doteqdot 1.84$ (2dp) ✓

Well done although some students could not handle the calculator

b) $$\frac{T_{k+1}}{T_k} = \frac{12 - k + 1}{k} \times \frac{3}{2}$$

if $\frac{T_{k+1}}{T_k} \geq 1$, $39 - 3k \geq 2k$

$5k \leq 39$

$k \leq 7\frac{4}{5}$

$k = 7$

$\therefore$ T_8 has greatest coefficient

$$T_8 = \binom{12}{7} (2x^2)^5 \left(\frac{3}{x}\right)^7.$$

Those who did not use the formula did not do as well.

Many students thought that T_8 implied $k = 8$?

$$T_8 = \binom{12}{7} 2^5 3^7 x^3$$

$\therefore$ greatest coefficient is $\binom{12}{7} 2^5 3^7$ ✓✓✓ | Binomial Theorem /

c) $V = \pi \int_0^{\pi} \sin^2 2x \, dx$, | Further Integration /3.

$$= \frac{\pi}{2} \int_0^{\pi} 1 - \cos 4x \, dx \cdot \quad ✓$$

Some could not make the appropriate substitution

$$= \frac{\pi}{2} \left[x - \frac{1}{4} \sin 4x \right]_0^{\pi} \quad ✓$$

$$= \frac{\pi}{2} \left[\left(\pi - \frac{1}{4} \sin 4\pi \right) - \left(0 - \frac{1}{4} \sin 0 \right) \right]$$

Generally well done.

$$= \frac{\pi}{2} \left(\pi - 0 - 0 + 0 \right)$$

Volume $= \frac{\pi^2}{2} u^3$. ✓

d) $y = \frac{x^2}{1-x^2}$

(i) if $1 - x^2 = 0$, $x = \pm 1$ these are the vertical asymptotes.

$$\lim_{x \to \infty} \frac{x^2}{1-x^2} = \lim_{x \to \infty} \frac{1}{\frac{1}{x^2} - 1} \quad ✓$$

$$= -1$$

$\therefore$ horizontal asymptote is $y = -1$ ✓

Students need clear setting out and descriptions of the processes to find the asymptotes.

(ii) $y' = \frac{(1-x^2)2x - x^2 \times -2x}{(1-x^2)^2}$

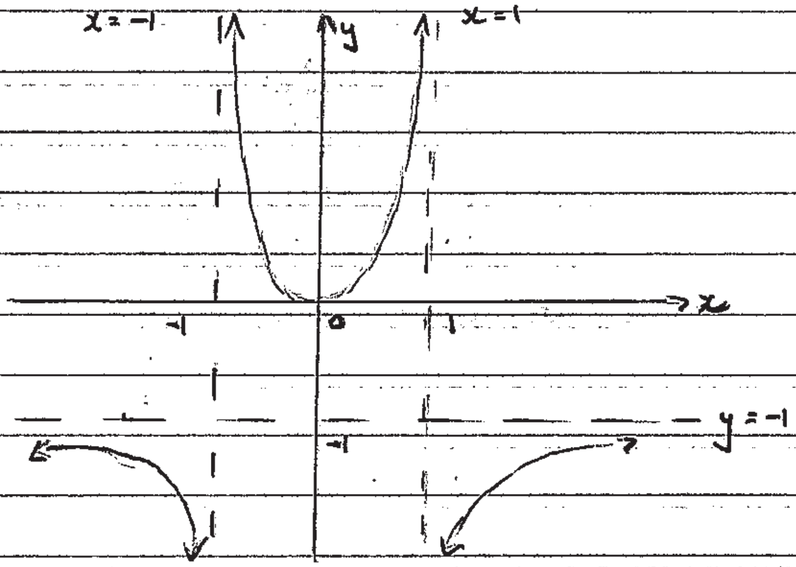
$$= \frac{2x - 2x^3 + 2x^3}{(1-x^2)^3}$$

$$= \frac{2x}{(1-x^2)^3}$$

Poor algebraic skills meant that several students could not handle the negatives.

if $y' = 0$, $x = 0$

$\therefore$ stationary point is $(0,0)$



Well done

10. a) $u = 1 + \tan\theta$

$du = \sec^2\theta \, d\theta$

if $\theta = \frac{\pi}{4}$, $u = 2$.

if $\theta = 0$, $u = 1$.

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta \, d\theta}{\sqrt{4 + 4\tan\theta}} = \int_1^2 \frac{du}{\sqrt{4u}}$$

$$= \frac{1}{2}\int_1^2 u^{-\frac{1}{2}} du$$

$$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_1^2$$

$$= \left[\sqrt{u}\right]_1^2$$

$$= \sqrt{2} - 1$$

Not well done by many students. Note the values of the limits, if $\theta = 0$ u is NOT 0.

The $\sqrt{4}$ caused most problems. Better to take the $\frac{1}{2}$ out before integrating.

b) (i) If $n = 1$, L.H.S. $= 1^2 = 1$

R.H.S. $= \frac{1}{6} \times 2 \times 3 = 1$

$\therefore$ true for $n = 1$

Assume true for $n = k$ (k integer $k \geq 2$)

i.e. Assume:

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k}{6}(k+1)(2k+1)$$

Well done.

Aim to prove true for $n = k+1$ using assumption.

i.e. $1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{k+1}{6}(k+2)(2k+3)$

LHS $= 1^2 + 2^2 + \dots + k^2 + (k+1)^2$

$= \frac{k}{6}(k+1)(2k+1) + (k+1)^2$ (using assumption)

$= (k+1)\left[\frac{k}{6}(2k+1) + k+1\right]$

$= \frac{k+1}{6}\left[k(2k+1) + 6(k+1)\right]$

expansion is a mistake. Find the common factor $k+1$.

$LHS = \frac{k+1}{6}(2k + 7k + 6)$

$= \frac{k+1}{6}(k+2)(2k+3)$

$= R.H.S.$

∴ Statement has been proved using the method of mathematical induction

(ii) $\frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^3} = \frac{\frac{n}{6}(n+1)(2n+1)}{n^3}$

$= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$\lim_{n\to\infty} \frac{1^2 + 2^2 + \cdots + n^2}{n^3} = \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$= \frac{1}{6} \times 1 \times 2$

$= \frac{1}{3}$

Students did not set this out in a logical fashion and mad errors as a result

c) (i) ∠AQB = ∠ABP (angle between tangent BP and chord equals angle in alternate segment)

∠APB = ∠BAQ (angle between tangent AQ and chord equals angle in alternate segment)

In Δ's AQB and PAB,

∠AQB = ∠ABP } proved above
∠BAQ = ∠APB }

∴ ΔAQB ||| ΔPAB (equiangular similarity test)

(ii) $\frac{QB}{AB} = \frac{AB}{AP}$ (sides in similar triangles are proportional)

∴ $AB^2 = PA \times BQ$

(iii) ∠ABQ = ∠APB (angles of similar triangles)

AP ∥ QB (as alternate angles are equal)

Some students did not choose angles in the triangles. This made the proof very long.

d) (i) Probability ripe $= \frac{p}{n}$

Probability not ripe $= 1 - \frac{p}{n} = \frac{n-p}{n}$

Probability at least one ripe

$$= 1 - P(\text{none ripe})$$

$$= 1 - \left(\frac{n-p}{n}\right)^2$$

$$= \frac{n^2 - (n-p)^2}{n^2}$$

$$= \frac{n^2 - n^2 + 2pn - p^2}{n^2}$$

$$= \frac{2pn - p^2}{n^2}$$

$$= \frac{p(2n-p)}{n^2}$$

(ii) Probability at least 2 are ripe

$$= 1 - P(\text{none ripe}) - P(1\text{ ripe})$$

$$= 1 - \left(\frac{n-p}{n}\right)^8 - \binom{8}{1}\left(\frac{n-p}{n}\right)^7 \times \frac{p}{n}$$

$$= 1 - \frac{(n-p)^8 + \binom{8}{1}(n-p)^7 \times p}{n^8}$$

$$= 1 - \frac{(n-p)^7\left[n-p+8p\right]}{n^8}$$

$$= 1 - \frac{(n-p)^7(n+7p)}{n^8}$$

Many students had run out of time for this question.

Be very careful of Algebra and watch negative signs.
This question contained some fudging.