THE SCOTS COLLEGE
Sydney

Extension One Mathematics

HSC Task 3

Weighting 20%

7th June 2011

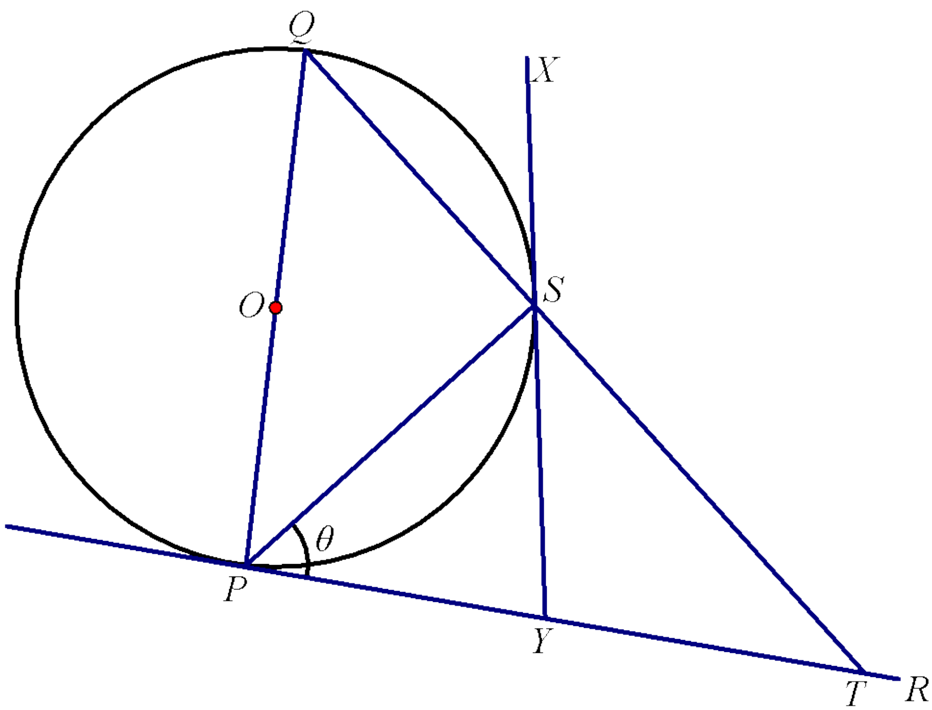
Total marks – 33

- Attempt all questions.

General Instructions

- **Working time – 45 Minutes**.
- Write using black or blue pen.
- Start a new page for every question
- Board-approved calculators may be used.
- All necessary working should be shown in every question.

QUESTION 1 **(8 Marks)**



PQ is the diameter of a circle with centre *O*. *PR* is the tangent to the circle at *P*. The line *QST* intersects the circle at *S* and *PR* at *T*. The tangent to the circle at *S* cuts *PR* at *Y*. Let .

a) **Prove** that (you are not allowed to use $PY = YS$). [3]

b) **Prove** that $PY = YS$. [2]

c) **Prove** that Y is the mid point of PT . [3]

QUESTION 2 (4 Marks) **START A NEW PAGE**

Prove the following by mathematical induction

is divisible by 9 for all $n \geq 1$.

QUESTION 3 (5 Marks) **START A NEW PAGE**

A vessel is in the shape of an inverted, right cone, with an apex angle of 60^{o}. Sand is being poured into the conical vessel at a constant rate of 48 cm^3/min.

a) Show that the radius of the sand in the cone at any time is — , where h is the height of the sand in the vessel. [2]

b) Find the rate at which the height of the sand in the vessel is increasing when the height is 12 *cm*. [3]

QUESTION 4 (8 Marks) **START A NEW PAGE**

An ice cube tray is filled with water at a temperature of 15^{o}C. It is then placed in a freezer which has a constant temperature of -10^{o}C. The water in the tray cools in the freezer at a rate that is proportional to the difference between the temperature of water, $W\ ^{0}$C, at any time, and the temperature of the freezer, $-10\ ^{o}$C, so that W satisfies the equation

— , where k is the constant of proportionality.

a) Show that ——— . [2]

b) Hence show that . [2]

c) After 5 minutes, the temperature of the water in the tray is $5\ ^{o}$C.

i. Find the value of k (to one decimal place) [2]

ii. Find the time that it takes for the water to cool down to $-5\ ^{o}$C (to the nearest minute). [2]

Question 5 continued on next page

<u>QUESTION 5</u> **(8 Marks)** **START A NEW PAGE**

Three mathematicians decide to invest $5000 into *PAL* Trust Fund in order to award an *annual* scholarship of $500 to a deserving Mathematics student. The investment earns an interest at the rate of 9% *p.a.*, compounded *monthly*. The scholarship amount of $500 is withdrawn at the end of each year after the monthly interest has been deposited.

a) Find the amount remaining in the investment at the end of the first year, after awarding the scholarship. [2]

b) Let A_n be the balance in the investment at the end of the n^{th} year, after the withdrawal of the scholarship amount for the year. If $R = 1.0075$, show that

[3]

c) Hence find the number of years for which the annual scholarship can be awarded (no scholarship is awarded if the balance in the investment falls below $500). [3]

END OF PAPER

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\, x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

EXTENSION 1 MATHEMATICS SOLUTIONS
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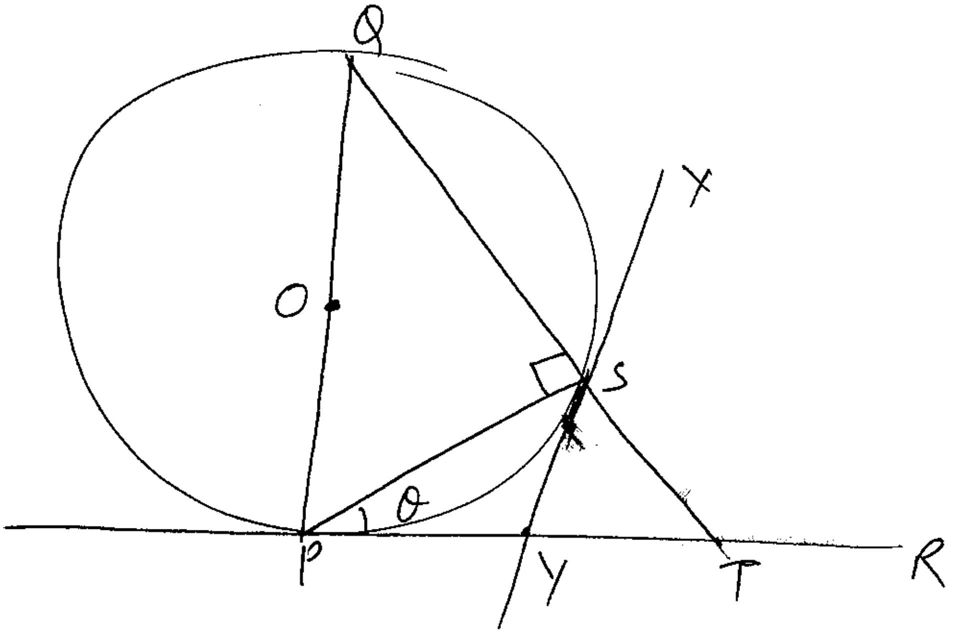
Question 3



(i) $\angle QPY = 90°$ (line from centre to tangent is $\perp$ at the point of contact)

$\therefore \angle QPS = 90 - \theta$

$\angle XSQ = \angle QPS$ (alternate segment theorem)

$= 90 - \theta$

$\angle YST = \angle XSQ$ (vertically opposite angle)

$\therefore \angle YST = 90 - \theta$.

(ii) $\angle QSP = 90°$ (angle in a semi-circle is a rt $\angle$)

$\therefore \angle PST = 90°$ (QST is a st. line)

$\therefore \angle PSY = \theta$ ($\because \angle YST = 90 - \theta$)

$= \angle SPY$

$\therefore PY = YS$ (equal sides opposite to equal angles of a Δ)

(iii) $\angle STP = 90 - \theta$ (angle sum of Δ PST)

$\therefore \angle YST = \angle STY$

$\therefore SY = YT$ (equal sides opposite to equal angles of a Δ)

$\therefore PY = YS = YT$

$\therefore$ Y is the mid pt of PT.

Q2:

$10^n + 3(4^{n+2}) + 5$ is divisible by 9 ~~for~~ all $n \geq 1$

Step 1 : prove true for $n = 1$

$10^1 + 3(4^{1+2}) + 5 = 10 + 3(64) + 5$

$= 207$

which is divisible by 9.

Step 2 Assume true for $n = k$

i.e. $10^k + 3(4^{k+2}) + 5 = 9p$ where p is a positive integer

Step 3 Prove true for $n = k+1$

i.e. $10^{k+1} + 3(4^{k+3}) + 5 = 9Q$ where Q is a positive integer.

LHS: $10 \cdot 10^k + 3(4 \cdot 4^{k+2}) + 5$

$= 10[10^k + 3(4^{k+2}) + 5] - 18(4^{k+2}) - 45$

$= 10[9P] - 18(4^{k+2}) - 45$

$= 9[10P - 2(4^{k+2}) - 5]$

$= 9Q$ where $Q = 10P - 2(4^{k+2}) - 5$ must be a positive integer as P and k are integers).

Hence true for $n = k+1$

Step 4 : Therefore by the principle of mathematical induction, it is true for all $n \geq 1$

Extension 1 Mathematics - Task 3
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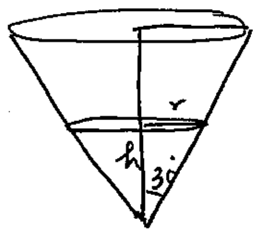
SOLUTIONS

Q. 3.
(a)



$\frac{dV}{dt} = 48 \text{ cm}^3/\text{min}$.

(i) $\frac{r}{h} = \tan 30°$

$r = h \tan 30°$

$r = \frac{h}{\sqrt{3}}$

(ii) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \frac{h^2}{3} . h$

$= \frac{\pi h^3}{9}$

$\frac{dV}{dh} = \frac{3\pi h^2}{9}$

$= \frac{\pi h^2}{3}$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$= 48 \times \frac{3}{\pi h^2}$

when $h = 12$ cm

$\frac{dh}{dt} = 48 \times \frac{3}{\pi (12)^2}$

$= \frac{1}{\pi}$ cm/min.

Question 4:

(i) $\frac{d}{dt}(We^{kt}) = e^{kt}\frac{dW}{dt} + W\frac{d}{dt}e^{kt}$

$= e^{kt}[-k(W+10)] + W \cdot ke^{kt}$

$= ke^{kt}[-W-10+W]$

$= -10ke^{kt}$ as required

(ii) $\therefore We^{kt} = -\int 10ke^{kt}$

$= -10e^{kt} + C$

When $t=0$, $W = 15$

$15 = -10 + C \qquad \therefore C = 25$

$\therefore We^{kt} = -10e^{kt} + 25 \qquad \div e^{kt}$

$W = -10 + 25e^{-kt}$

or $W = 25e^{-kt} - 10$.

(iii) When $t = 5$ min, $W = 5°C$

(a) $5 = 25e^{-5k} - 10$

$e^{-5k} = \frac{15}{25} = 3/5$

$k = \frac{\ln 3/5}{-5} = 0.1$ (1 dp)

(b) When $W = -5°C$

$-5 = 25e^{-kt} - 10$

$5 = 25e^{-kt}$

$e^{-kt} = 1/5$

$t = \frac{\ln 1/5}{-k} = 15.75\ldots$

$\simeq 16$ min

Q.5 $P = \$5000$

$r = 9\%$ pa $= 0.75\%$ p. month

$= 0.0075$

$M = \$500$

(i) $A_1 = 5000(1.0075)^{12} - 500$

$= \$4969.03$.

(ii) $A_2 = [5000(1.0075)^{12} - 500](1.0075)^{12} - 500$

$= 5000(1.0075)^{24} - 500(1.0075)^{12} - 500$

$A_3 = 5000(1.0075)^{36} - 500(1.0075)^{24} - 500(1.0075)^{12} - 500$

⋮

$A_n = 5000(1.0075)^{12n} - 500(1.0075)^{12(n-1)} - \dots - 500(1.0075)^{12} - 500$

$= 5000(1.0075)^{12n} - 500[1 + 1.0075^{12} + 1.0075^{24} + \dots + 1.0075^{12(n-1)}]$

$= 5000(1.0075)^{12n} - 500\left[\frac{1(1.0075^{12n} - 1)}{1.0075^{12} - 1}\right]$

$= 5000 R^{12n} - \frac{500[R^{12n} - 1]}{R - 1}$

(iii) $A_n = 0$

$0 = 10(1.0075)^{12n} - \frac{1}{1.0075^{12} - 1}(1.0075)^{12n} + \frac{1}{1.0075^{12} - 1}$

$(1.0075)^{12n}\left(\frac{1}{1.0075^{12} - 1} - 10\right) = \frac{1}{(1.0075)^{12} - 1}$

$1.0075^{12n} = \frac{\frac{1}{(1.0075)^{12} - 1}}{\frac{1}{(1.0075)^{12} - 1} - 10} = x$

$12n = \frac{\ln x}{\ln 1.0075}$ $\therefore n = \frac{1}{12}\frac{\ln x}{\ln 1.0075}$

$= 31.02\dots$

≈ 31 years.