THE SCOTS COLLEGE

MATHEMATICS EXTENSION I

YEAR 12 ASSESSMENT TASK 3

4TH JUNE 2012

GENERAL INSTRUCTIONS

- Working time – 45 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- Show all necessary working
- *Each question must be completed in a new answer booklet.*
- Label each answer booklet with your name, class teacher's name and the question number attempted

WEIGHTING

20%

TOTAL MARKS

32

QUESTION 1 - ANSWER IN A NEW BOOKLET 10 MARKS

a) Use the principle of mathematical induction to show that $8^n - 7n + 6$ is divisible by 7 for all positive integral values of *n*. **4 marks**

b) A certain rectangle, with its length double its width, is also a cyclic quadrilateral.

i. Let the length of the rectangle be *y*. **1 mark**

Draw a diagram of the given information.

ii. Show that the radius of the circle, r, in terms of y is $\frac{y}{4}\sqrt{5}$ **2 marks**

iii. If the length of the rectangle, *y*, is decreasing at a rate of 3cm per second, calculate the rate the area of the circle is changing when the length of the rectangle is 4cm **3 marks**

QUESTION 2 - ANSWER IN A NEW BOOKLET 17 MARKS

a) A particle in simple harmonic motion starts from a displacement of -6 units and takes eight seconds to travel to a displacement of 2 units. The particle oscillates between -6 units and 2 units.

i. State the period of the particle **1 mark**

ii. Sketch the displacement–time graph for $0 \leq t \leq 24$ **3 marks**

iii. State the amplitude and centre of motion of the particle **2 marks**

iv. Write down an equation for the position of the particle *x* at time *t* seconds in the form $x = a\cos(nt) + b$ where *a, b* and *n* are constants **2 marks**

v. Hence, prove the particle moves in simple harmonic motion **2 marks**

QUESTION 2 CONTINUES ON THE NEXT PAGE

b) The motion of a particle is described by the acceleration equation

$$\frac{d^2x}{dt^2} = x$$

where *x* is the displacement of the particle from the origin in metres and *t* is time in seconds

i. Initially the particle is at the origin travelling with a velocity of -3m/s. **3 marks**

Show that the velocity of the particle, in terms of x, is: $v = -\sqrt{x^2 + 9}$

ii. By using the Standard Integrals Table, evaluate $\frac{d}{dx}\ln\left(x + \sqrt{x^2 + 9}\right)$ **1 mark**

iii. Hence, or otherwise, find a time when the particle is 4 metres to the left of the origin. Give your answer to the nearest second. **3 marks**

QUESTION 3 – ANSWER IN A NEW BOOKLET 6 MARKS

a) Morgan deposits $20,000 in a savings account with 6% per annum interest calculated at the end of each month.

i. Write an expression for the value of Morgan's deposit at the end of the third month. **1 mark**

ii. From the fourth month Morgan decides to withdraw $800 just after interest is calculated and he continues to withdraw $800 every month thereafter. **3 marks**

Let B_n be the balance of Morgan's savings account at the end of the n^{th} $800 withdrawal.

Show that $B_n = 20000(1.005)^{n+3} - 160000(1.005^n - 1)$

iii. Hence, how many times can Morgan withdraw $800 from his account? **2 marks**

END OF EXAM

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Q1) a) ① Prove true for $n=1$

$8-7+6=7$

Divisible by 7

$\therefore$ true for $n=1$

② Assume true for $n=k$:

$8^k-7k+6=7M$ where M is an integer.

③ Prove true for $n=k+1$

$8^{k+1}-7(k+1)+6$

$(8^k)8-7k-7+6$ $\qquad 8^k=7M+7k-6$

$(8^k)8-7k-1$

$(7M+7k-6)8-7k-1$

$56M+56k-48-7k-1$

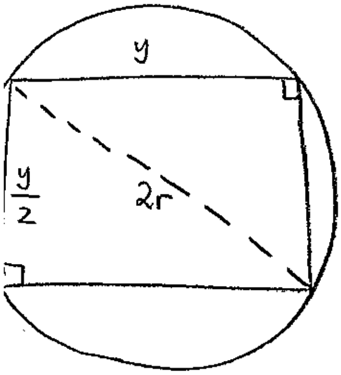
$56M+49k-49$

$7(8M+7k-7)$

$\therefore$ Divisible by 7

True for $n=k+1$

④ Since true for $n=1$ and $n=k+1$, then true for $n=2$. Since true for $n=2$ and $n=k+1$ then true for $n=3$ and so forth for all positive values of n.



ii) $y^2 + \left(\frac{y}{2}\right)^2 = (2r)^2$

$y^2 + \frac{y^2}{4} = 4r^2$

$\frac{5y^2}{4} = 4r^2$

$r = \sqrt{\frac{5y^2}{16}} = \frac{y}{4}\sqrt{5}$

$A = \pi r^2$

$A = \pi\left(\frac{5y^2}{16}\right)$

$\frac{dA}{dy} = \frac{5\pi y}{8}$

$= -3$ cm/s

$= \frac{dy}{dt} \times \frac{dA}{dy}$

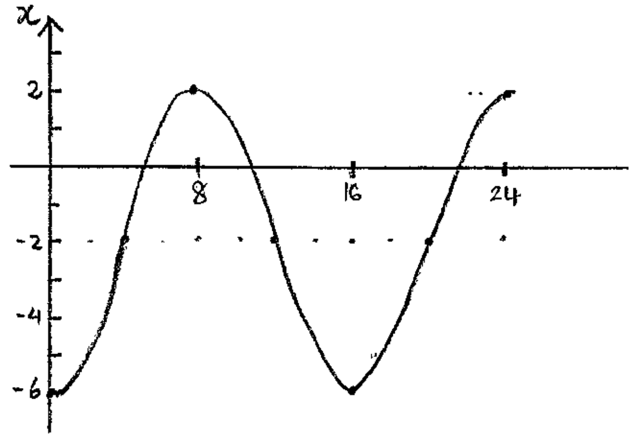
$= -3 \times \frac{5\pi y}{8}$

$y = 4$.

$= \frac{-15\pi}{2}$ cm^2/s

Q2) a) i) Period = 16 seconds.

ii)



iii) Amplitude = 4

Centre of motion = −2

iv) $x = a\cos(nt) + b$

$x = 4\cos(nt) - 2$

$16 = \frac{2\pi}{n} \quad \therefore n = \frac{2\pi}{16} = \frac{\pi}{8}$

$\therefore x = 4\cos\left(\frac{\pi}{8}t\right) - 2$

v) $v = -\frac{4\pi}{8}\sin\left(\frac{\pi}{8}t\right)$

$= -\frac{\pi}{2}\sin\left(\frac{\pi}{8}t\right)$

$a = -\frac{\pi^2}{16}\cos\left(\frac{\pi}{8}t\right)$

Note: $\frac{x+2}{4} = \cos\left(\frac{\pi}{8}t\right)$

$a = -\frac{\pi^2}{64}(x+2)$

This is in the form $a = -n^2(x+A)$

∴ SHM.

Q2) b) $a = x$

i) $t = 0, x = 0, v = -3$

$\frac{d}{dx}\frac{1}{2}v^2 = x$

$\frac{1}{2}v^2 = \frac{x^2}{2} + C_1$

$\frac{1}{2}(9) = 0 + C_1$

$C_1 = \frac{9}{2}$

$\frac{1}{2}v^2 = \frac{x^2}{2} + \frac{9}{2}$

$v^2 = x^2 + 9$

$v = \pm\sqrt{x^2+9}$

When $x = 0, v = -3$

$\therefore v = -\sqrt{x^2+9}$

ii) $\frac{d}{dx}\ln\left(x + \sqrt{x^2+9}\right)$

$= \frac{1}{\sqrt{x^2+3^2}} = \frac{1}{\sqrt{x^2+9}}$

iii) $t = ?$ when $x = -4$

$v = -\sqrt{x^2+9}$

$\frac{dx}{dt} = -\sqrt{x^2+9} \quad \therefore \quad dt = \int \frac{-1}{\sqrt{x^2+9}}\,dx$

$t = \int \frac{-1}{\sqrt{x^2+9}}\,dx$

$t = -\ln\left(x + \sqrt{x^2+9}\right) + C_2$

$0 = -\ln\left(0 + \sqrt{9}\right) + C_2$

$C_2 = \ln(3)$

$\therefore t = -\ln\left(x + \sqrt{x^2+9}\right) + \ln(3)$

when $x = -4, t = ?$

$t = -\ln\left(-4 + \sqrt{16+9}\right) + \ln 3$

$= -\ln(1) + \ln 3$

$t = \ln 3$

≈ 1 second

Q3 a) i) $20000(1.005)^3$

ii) $B_1 = 20000(1.005)^4 - 800$

$B_2 = 20000(1.005)^5 - 800(1.005) - 800$

$B_3 = 20000(1.005)^6 - 800(1.005)^2 - 800(1.005) - 800$

$B_4 = 20000(1.005)^7 - 800(1.005)^3 - 800(1.005)^2 - 800(1.005) - 800$

$= 20000(1.005)^7 - 800[1.005^3 + 1.005^2 + 1.005 + 1]$

$\therefore B_n = 20000(1.005)^{n+3} - 800[1.005^{n-1} + 1.005^{n-2} + \ldots + 1.005 + 1]$

$= 20000(1.005)^{n+3} - 800\left[1\frac{(1.005^n - 1)}{1.005 - 1}\right]$

$B_n = 20000(1.005)^{n+3} - 160000(1.005^n - 1)$

iii) $B_n = 20000(1.005)^n(1.005)^3 - 160000(1.005)^n + 160000 = 0$

$= 4000[(1.005)^n(5(1.005)^3 - 40) + 40] = 0$

$(1.005)^n = \frac{-40}{5(1.005)^3 - 40}$

$n = \ln\left(\frac{-40}{5(1.005)^3 - 40}\right) \div \ln(1.005)$

$= 27.205$

$\therefore$ Morgan can withdraw \$800 27 times.