St George Girls High School

Year 12

Assessment Task 3

2006

Mathematics Extension 1

General Instructions

- Time allowed – 75 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question on a new page.

Total marks – 70

- Attempt Questions 1 – 5
- All questions are of equal value

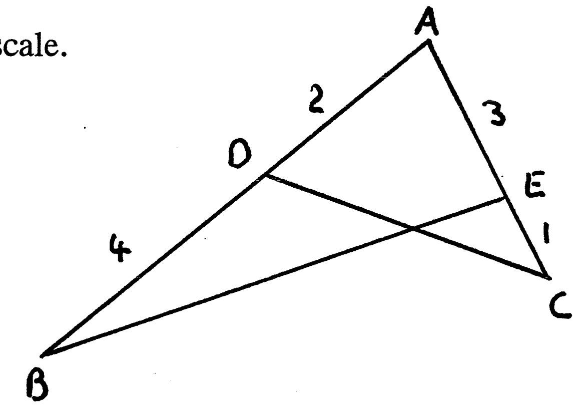
Question	Mark
Question 1	/14
Question 2	/14
Question 3	/14
Question 4	/14
Question 5	/14
Total	/70

<u>Question 1</u> – (14 marks) – Start a new page **Marks**

a) Find the value of 'a' such that $P(x) = x^3 - 2x^2 - ax + 6$ is divisible by $(x+2)$ 2

b) The diagram opposite is <u>not</u> drawn to scale. 4

Given $AD = 2;\ BD = 4$
$AE = 3;\ CE = 1$



(i) <u>Prove</u> that $\Delta ABE \parallel \Delta ACD$

(ii) If $BE = 5$, find the length of DC

c) Find $\int_0^{\frac{\pi}{2}} \cos^2 \frac{x}{2}\, dx$ 3

d) If α, β, γ are the roots of the cubic equation $5x^3 - 2x^2 - 3 = 0$, 5
find the value of:

(i) $\alpha + \beta + \gamma$

(ii) $a\beta + \alpha\gamma + \beta\gamma$

(iii) $\alpha\beta\gamma$

(iv) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

Marks

<u>Question 2</u> – (14 marks) – Start a new page

a) (i) Show that $\sqrt{12}\sin x + 2\cos x \equiv 4\cos\left(x - \frac{\pi}{3}\right)$ 5

(ii) Hence, solve the equation $\sqrt{12}\sin x + 2\cos x = -3$ for $0 \leq x \leq 2\pi$

[Give all answers correct to two decimal places]

b) (i) Factorise completely the polynomial $P(x) = x^3 - x^2 - 8x + 12$ given that the equation $P(x) = 0$ has a double root at $x = 2$ 3

(ii) When $Q(x) = ax^3 + bx + c$ is divided by $(x-1)$, the remainder is -4.

When $Q(x) = ax^3 + bx + c$ is divided by $(x^2 - 4)$, the remainder is $(-4x+3)$ 4

Find a, b and c.

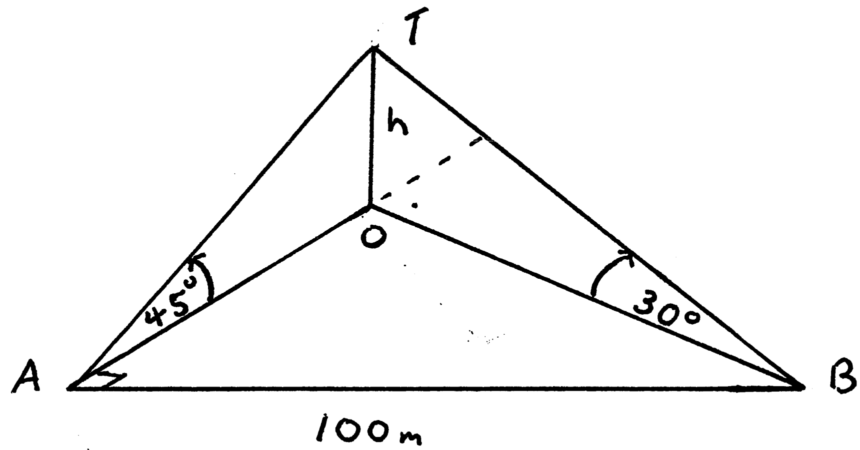
c) A particle moving in a straight line has its acceleration as a function of time given by $\ddot{x} = 3t - 1$

If the particle is initially at rest two units to the right of the origin, find its displacement at $t = 3$ 2

Question 3 – (14 marks) – Start a new page **Marks**

a) Prove $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$ 2

b) 6



A surveyor stands at a point A, which is due south of a tower OT of height h metres. The angle of elevation of the top of the tower from A is 45°. The surveyor then walks 100m due east to point B, from where she measures the angle of elevation of the top of the tower to be 30°.

(i) Express the length OB in terms of h.

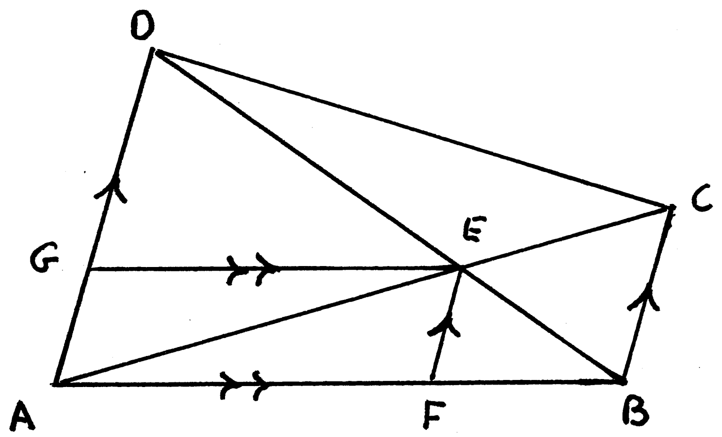
(ii) Show that $h = 50\sqrt{2}$

(iii) Calculate the bearing of B from the base of the tower.

Question 3c) on next page

Question 3 (cont'd) **Marks**

c) 6



$ABCD$ is a trapezium in which

(i) $BC \parallel AD$ (ii) $BC : AD = 1:2$

(iii) $EF \parallel BC$ (iv) $EG \parallel AF$

(α) Show that $FE : BC = AF : AB$

(β) Show that $DG : AD = GE : AB$

(γ) Show that $FE : BC = DG : AD$

(ε) Show that $DG = AD - FE$

Hence, (or otherwise) prove that the ratio $FE : BC = 2:3$

Question 4 – (14 marks) – Start a new page **Marks**

a) The polynomials $4x^3 - x^2 + 3$ and $ax(x+1)(x+2) + bx(x+1) + cx + d$ are equal for 4 values of x; determine the values of a, b, c and d. 4

b) The velocity $\dot{x}\, ms^{-1}$ of a particle moving in a straight line is given by $\dot{x} = \frac{4}{\pi}\sin\frac{\pi t}{2}$ 7

Initially the particle is at rest at the origin.

(i) Find an expression for displacement as a function of time.

(ii) Find an expression for acceleration as a function of time.

(iii) Sketch the graph of $\dot{x}$ as a function of time over $0 \le t \le 8$.

(iv) What is the displacement of the particle each time it is at rest during the period $0 \le t \le 8$.

c) Show that $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ 3

Hence, find $\int 12\sin 4x\cos 2x\, dx$

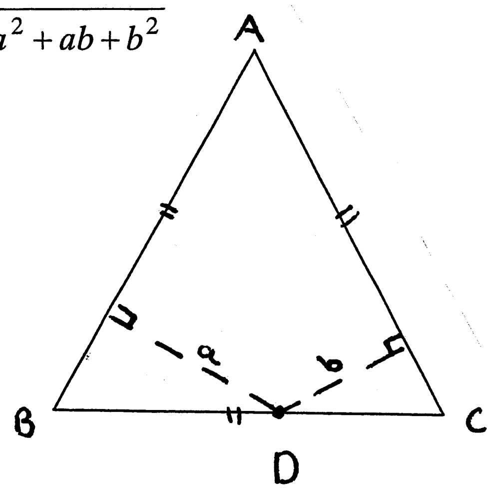
Question 5 – (14 marks) – Start a new page **Marks**

a) (i) Write down an expression for $\sin(x - y)$. 6

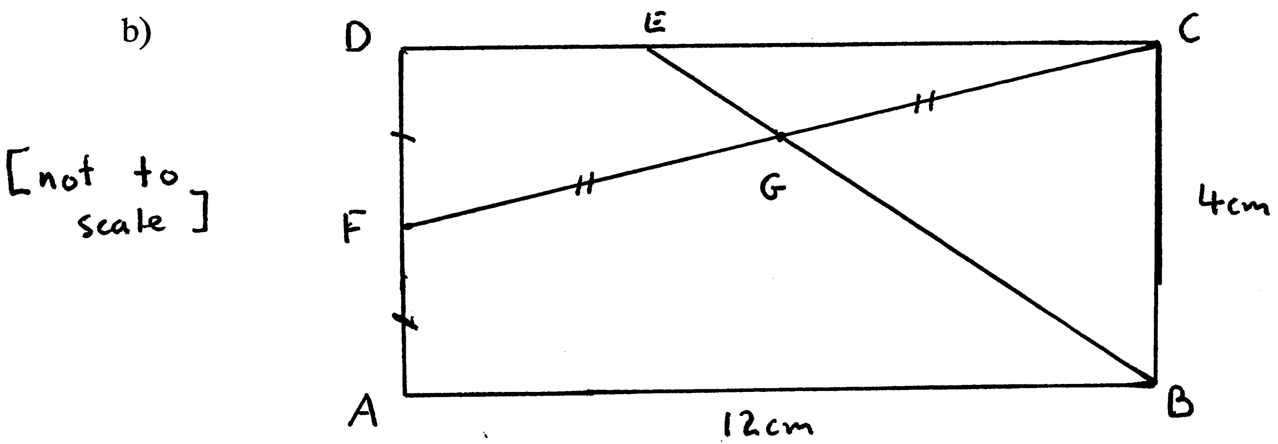
(ii) Given $\sin\alpha = c$ and $\sin(60 - \alpha) = d$ prove that $c^2 + cd + d^2 = \frac{3}{4}$

(iii) ΔABC is equilateral and D is any point on the side BC. The lengths of the perpendiculars from D to AB and AC are a and b respectively.

Prove that $AD = \frac{2}{\sqrt{3}}\sqrt{a^2 + ab + b^2}$



b) 5



In the figure, $ABCD$ is a rectangle. $AB = 12\text{cm}$ and $BC = 4\text{cm}$
F is the midpoint of AD and G is the midpoint of CF.

Find the length of DE. [Hint: Construct perpendicular from G to X on DC].

c) If $a\cos x = 1 + \sin x$ prove that $\frac{a-1}{a+1} = t$, where $t = \tan\frac{x}{2}$ 3

End of Paper

TABLE OF STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

QUESTION 1:

(a) $P(x) = x^3 - 2x^2 - ax + 6$

$P(-2) = 0 \Rightarrow -8 - 8 + 2a + 6 = 0$

$2a = 10$

$\therefore a = 5$

(b) (i) In ΔABE, ΔACD

(i) $\hat{A}$ is common

(ii) $\frac{AB}{AC} = \frac{3}{2} = \frac{AE}{AD}$

$\therefore \Delta ABE \;|||\; \Delta ACD$ (one angle equal and the sides about that angle are in same ratio)

(ii) $\frac{5}{x} = \frac{3}{2} \Rightarrow 3x = 10$

$x = \frac{10}{3}$

(c) $\int_0^{\frac{\pi}{2}} \cos^2\left(\frac{x}{2}\right) dx = \int_0^{\frac{\pi}{2}} \frac{\cos x + 1}{2} dx$

$= \frac{1}{2}\left[\sin x + x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[1 + \frac{\pi}{2} - 0\right]$

$= \frac{\pi + 2}{4}$

(d) (i) $\alpha + \beta + \gamma = \frac{2}{5}$ (ii) $\alpha\beta + \alpha\gamma + \beta\gamma = 0$

(iii) $\alpha\beta\gamma = \frac{3}{5}$ (iv) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$

$= 0$

QUESTION 2:

(a) (i) $4\cos\left(x - \frac{\pi}{3}\right) = 4\cos x \cos\frac{\pi}{3} + 4\sin x \sin\frac{\pi}{3}$

$= 2\cos x + 4\sin x \cdot \frac{\sqrt{3}}{2}$

$= 2\sqrt{3}\sin x + 2\cos x$

$= \sqrt{12}\sin x + 2\cos x.$

(ii) $4\cos\left(x - \frac{\pi}{3}\right) = -3$

$\Rightarrow \cos\left(x - \frac{\pi}{3}\right) = -\frac{3}{4}$

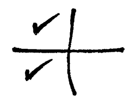
$0 \le x \le 2\pi$

$-\frac{\pi}{3} \le x - \frac{\pi}{3} \le \frac{5\pi}{3}$

$\therefore x - \frac{\pi}{3} = \pi - \alpha, \pi + \alpha$

$x = \frac{4\pi}{3} - \alpha, \frac{4\pi}{3} + \alpha$

$\left(x - \frac{\pi}{3}\right)_{acute} = 0.7227\ldots = \alpha$



$= 3.47, 4.91$

(b) (i) $P(x) = x^3 - x^2 - 8x + 12$

Let roots of $P(x) = 0$ be $2, 2, \alpha$

$\sum$ roots $= 1 \Rightarrow 4 + \alpha = 1$

$\alpha = -3$

$\therefore P(x) = (x-2)(x-2)(x+3)$

(ii) $Q(x) = ax^3 + bx + c$

$Q(1) = -4 \Rightarrow a + b + c = -4$ ——— (1)

$Q(x) = (x^2 - 4).P(x) + (-4x + 3)$

$\therefore Q(2) = -5 \Rightarrow 8a + 2b + c = -5$ ——— (2)

$Q(-2) = 11 \Rightarrow -8a - 2b + c = 11$ ——— (3)

(2) + (3): $2c = 6$

$c = 3$ sub in (1), (2)

$\Rightarrow a + b = -7$ ——— (4)

$8a + 2b = -8$ ——— (5)

(5) − 2 × (4): $6a = 6$

(c) $\ddot{x} = 3t - 1$

$$\Rightarrow \frac{dv}{dt} = 3t - 1$$

$$\therefore v = \frac{3t^2}{2} - t + c$$

$$\left.\begin{matrix} t=0 \\ v=0 \end{matrix}\right\} \Rightarrow 0 = 0 + c \qquad \therefore c = 0$$

$$\therefore \frac{dx}{dt} = \frac{3t^2}{2} - t$$

$$\therefore x = \frac{t^3}{2} - \frac{t^2}{2} + c$$

$$\left.\begin{matrix} t=0 \\ x=2 \end{matrix}\right\} \Rightarrow 2 = c$$

$$\therefore x = \frac{t^3}{2} - \frac{t^2}{2} + 2$$

at $t=3$ $\quad x = \frac{27}{2} - \frac{9}{2} + 2$

$$= 11$$

QUESTION 3:

(a) $LHS = \dfrac{\sin 2x}{1+\cos 2x}$

$= \dfrac{2\sin x \cos x}{1+(2\cos^2 x - 1)}$

$= \dfrac{2\sin x \cos x}{2\cos^2 x}$

$= \tan x$

$= RHS$

$\therefore \dfrac{\sin 2x}{1+\cos 2x} = \tan x$

(b) (i) In ΔOBT: $\tan 60^\circ = \dfrac{OB}{h}$

$\therefore OB = h\tan 60^\circ$

$= h\sqrt{3}$

(ii) In ΔOAT: $OA = h$ (isosceles ΔOAT, $O\hat{A}T = O\hat{T}A = 45^\circ$)

In ΔOAB: $OB^2 = OA^2 + 100^2$

$\therefore (h\sqrt{3})^2 = h^2 + 100^2$

$3h^2 = h^2 + 10\,000$

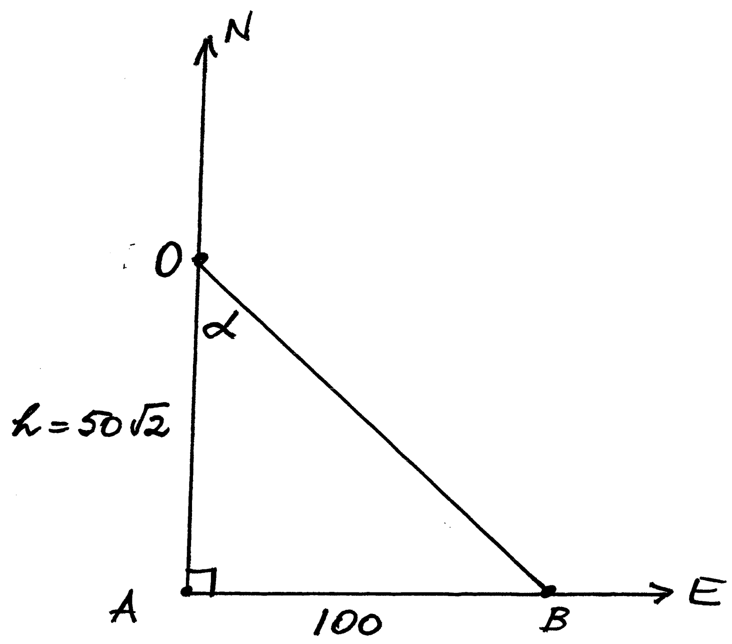
$2h^2 = 10\,000$

$h^2 = 5\,000$

$h = \sqrt{5000}$

$= 50\sqrt{2}$

(iii) Birds eye view



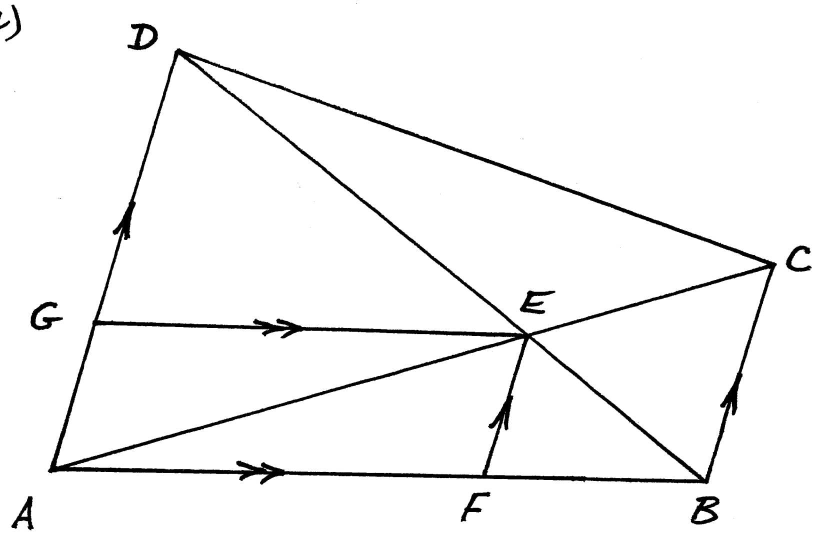
In Δ OAB: $\tan A\hat{O}B = \dfrac{100}{50\sqrt{2}}$

$= \sqrt{2}$

$\therefore A\hat{O}B = 54^\circ 44'$

$\therefore$ Bearing of B from base of tower is $125^\circ 16'$

(c)



(c)

(α) In ΔAEF, ΔACB

- $\hat{A}$ is common
- $A\hat{F}E = A\hat{B}C$ (corresponding angles equal, $FE \parallel BC$)
- $A\hat{E}F = A\hat{C}B$ (... . . . .)

$\therefore \Delta AEF \parallel\!| \Delta ACB$ (equiangular)

$\therefore FE : BC = AF : AB$ (corresponding sides of similar triangles in same ratio)

(β) Similarly $\Delta DGE \parallel\!| \Delta DAB$

$\therefore DG : AD = GE : AB$ (corresponding sides of similar triangles in same ratio)

(γ) now $\dfrac{FE}{BC} = \dfrac{AF}{AB}$

$= \dfrac{GE}{AB}$ since $AF = GE$ (opposite sides of parallelogram are equal)

$= \dfrac{DG}{AD}$ from (β)

QUESTION 4:

(a) $4x^3 - x^2 + 3 \equiv ax(x+1)(x+2) + bx(x+1) + cx + d$

since the two polynomials are equal for 4 values of x.

$x = 0 \Rightarrow 3 = d$

$x = -1 \Rightarrow -2 = -c + d$

$= -c + 3$

$\therefore c = 5$

coeff of $x^3 \Rightarrow 4 = a$

$x = -2 \Rightarrow -33 = -2b(-1) - 2c + d$

$-33 = 2b - 10 + 3$

$\therefore 2b = -26$

$\therefore b = -13$

$\therefore a = 4, b = -13, c = 5, d = 3$

(b) $\dot{x} = \frac{4}{\pi} \sin \frac{\pi t}{2}$

(i) $x = -\frac{4}{\pi} \cdot \frac{\cos \frac{\pi t}{2}}{\left(\frac{\pi}{2}\right)} + C$

$x = -\frac{8}{\pi^2} \cos \frac{\pi t}{2} + C$

at $t = 0, x = 0 \quad \therefore 0 = -\frac{8}{\pi^2} + C$

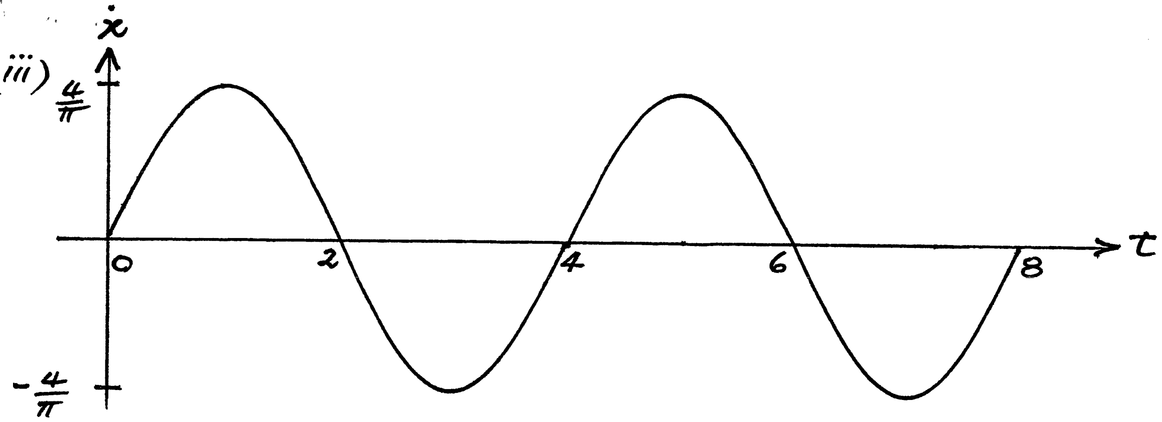
$\therefore C = \frac{8}{\pi^2}$

$\therefore x = \frac{8}{\pi^2} - \frac{8}{\pi^2} \cos \frac{\pi t}{2}$ ———— (1)

(ii) $\ddot{x} = \frac{4}{\pi} \cdot \frac{\pi}{2} \cos \frac{\pi t}{2}$

$= 2 \cos \frac{\pi t}{2}$

(iii)



(iv) Particle at rest at $\dot{x} = 0$

ie $t = 0, 2, 4, 6, 8$ sub in ①

$\left.\begin{matrix} t=0 \\ x=0 \end{matrix}\right\}$ $\left.\begin{matrix} t=2 \\ x=\frac{16}{\pi^2} \end{matrix}\right\}$ $\left.\begin{matrix} t=4 \\ x=0 \end{matrix}\right\}$ $\left.\begin{matrix} t=6 \\ x=\frac{16}{\pi^2} \end{matrix}\right\}$ $\left.\begin{matrix} t=8 \\ x=0 \end{matrix}\right\}$

(c) $\sin(A+B) + \sin(A-B)$

$= \sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$

$= 2 \sin A \cos B$ ——— ①

$\int 12 \sin 4x \cos 2x \, dx$

$= 6 \int 2 \sin 4x \cos 2x \, dx$

$= 6 \int (\sin 6x + \sin 2x) \, dx$ from ①

$= 6 . \left[-\frac{1}{6} \cos 6x - \frac{1}{2} \cos 2x \right] + C$

$= -\cos 6x - 3 \cos 2x + C$

QUESTION 5 :

(a) (i) $\sin(x-y) = \sin x \cos y - \cos x \sin y$

(ii) $\left.\begin{aligned} \sin\alpha &= c \\ \sin(60^\circ-\alpha) &= d \end{aligned}\right\} \quad \begin{aligned} &\text{——— (1)} \\ &\text{——— (2)} \end{aligned}$

from (2): $d = \sin 60^\circ \cos\alpha - \cos 60^\circ \sin\alpha$

$$= \frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha$$

$$d = \frac{\sqrt{3}}{2}\cos\alpha - \frac{c}{2}$$

$$\Rightarrow \quad 2d = \sqrt{3}\cos\alpha - c$$

$$\Rightarrow \quad \cos\alpha = \frac{2d+c}{\sqrt{3}}$$

Then $\sin^2\alpha + \cos^2\alpha = 1$

$$\Rightarrow \quad c^2 + \left(\frac{2d+c}{\sqrt{3}}\right)^2 = 1$$

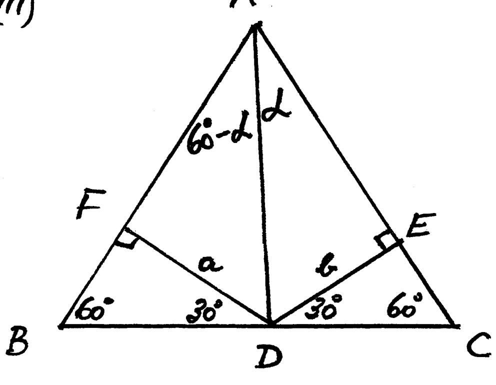
$$c^2 + \frac{4d^2+4cd+c^2}{3} = 1$$

$$3c^2 + 4d^2 + 4cd + c^2 = 3$$

$$4c^2 + 4cd + 4d^2 = 3$$

$$\therefore \; c^2 + cd + d^2 = \frac{3}{4}$$

(iii)



Let $D\hat{A}C = \alpha$

$D\hat{A}B = 60^\circ - \alpha$

In ΔADE: $\sin\alpha = \dfrac{b}{AD}$

ΔADF: $\sin(60^\circ - \alpha) = \dfrac{a}{AD}$

(ii) let $c = \frac{b}{AD}$ and $d = \frac{a}{AD}$

Then $c^2 + cd + d^2 = \frac{3}{4}$

$$\Rightarrow \frac{b^2}{AD^2} + \frac{ab}{AD^2} + \frac{a^2}{AD^2} = \frac{3}{4}$$

$$\Rightarrow b^2 + ab + a^2 = \frac{3}{4} . AD^2$$

$$\therefore AD = \frac{2}{\sqrt{3}}(b^2 + ab + a^2)^{\frac{1}{2}}$$

b) Construct $GX \perp DC$ at X.

$CX = XD = 6$ cm (parallel lines AD, GX and BC cut intercepts in same ratio and $FG = GC$)

$GX = \frac{1}{2} DF$ (The line joining the mid-points of two sides of ΔCDF is parallel to the third side and equal to half its length)

$= 1$

OR prove the same by similar triangles.

Let $DE = x$

Hence $EX = 6 - x$

In ΔEXG, ΔECB

(i) $\hat{E}$ is common

(ii) $E\hat{X}G = E\hat{C}B = 90°$ (by construction and angles of a rectangle)

(iii) $E\hat{G}X = E\hat{B}C$ (angle sum of triangle is 180°)

$\therefore \Delta EXG ||| \Delta ECB$ (equiangular)

$$\therefore \frac{EX}{EC} = \frac{XG}{CB}$$ (corresponding sides of similar triangles are in the same ratio)

$$\frac{6-x}{12-x} = \frac{1}{4}$$

$\Rightarrow x = 4$ $\quad \therefore DE = 4$ cm

(c) $a\cos x = 1 + \sin x$

$$\Rightarrow \quad a = \sec x + \tan x$$

$$\frac{a-1}{a+1} = \frac{\sec x + \tan x - 1}{\sec x + \tan x + 1}$$

$$= \frac{1 + \sin x - \cos x}{1 + \sin x + \cos x}$$

$$= \frac{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \qquad t = \tan\frac{x}{2}$$

$$= \frac{1+t^2 + 2t - 1 + t^2}{1+t^2 + 2t + 1 - t^2}$$

$$= \frac{2t^2 + 2t}{2t + 2}$$

$$= \frac{2t(t+1)}{2(t+1)}$$

$$= t \qquad \text{where } t = \tan\frac{x}{2}$$