St George Girls High School

Year 12

Assessment Task 3

2007

Mathematics Extension 1

General Instructions

- Time allowed – 75 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question on a new page.

Total marks – 70

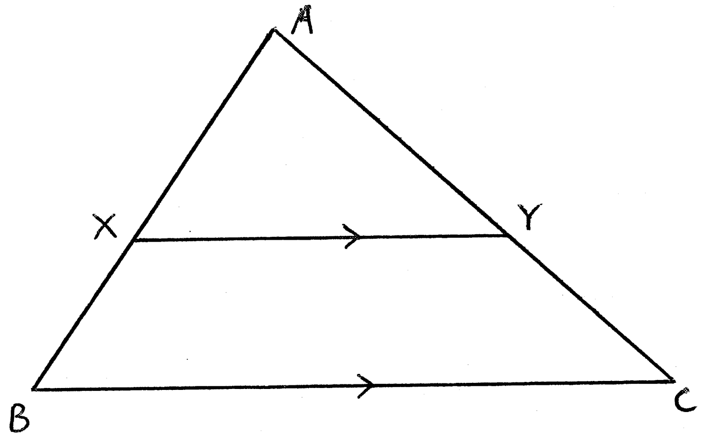
- Attempt Questions 1 – 5
- All questions are of equal value

Question	Mark
Question 1	/14
Question 2	/14
Question 3	/14
Question 4	/14
Question 5	/14
Total	/70

Question 1 – (14 marks) – Start a new page **Marks**

a) Differentiate $x\tan^{-1}\dfrac{x}{2}$ 3

b)



In the triangle ABC, $XY = 8\text{cm}$, $BC = 14\text{cm}$, $AC = 18\text{cm}$ and $XY \parallel BC$.

(i) Prove that ΔAXY is similar to ΔABC 3

(ii) Find the length of AY giving reasons. 3

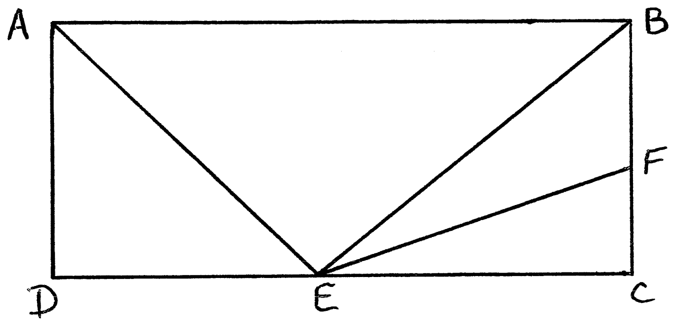
c) For the function $y = \sin^{-1}\left(\dfrac{x}{2}\right)$:

(i) State the domain and range 2

(ii) Sketch the graph of the function 3

Question 2 – (14 marks) – Start a new page **Marks**

a)



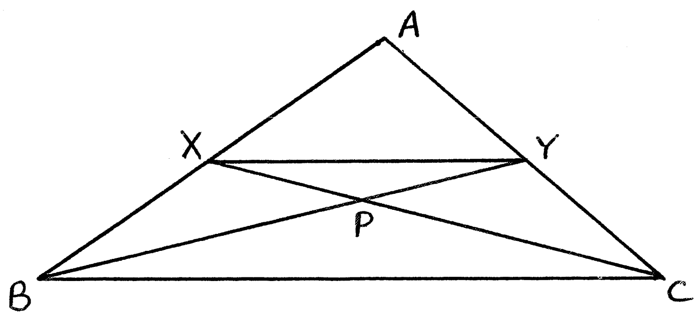
$ABCD$ is a rectangle. E is a point on DC such that AE bisects angle DEB and EF bisects angle BEC. Prove that angle $AEF = 90°$ 3

b) Evaluate $\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx$ 2

c) Find the value of the expression

$\cos^{-1}\left(-\frac{1}{2}\right)-\sin^{-1}\left(-\frac{1}{2}\right)$ in terms of π 2

d) In the diagram below, $\angle AXY = \angle AYX$ and $XP = YP$



(i) Copy this diagram on your sheet. 1

(ii) Prove that $\Delta ABY \equiv \Delta ACX$, giving reasons. 4

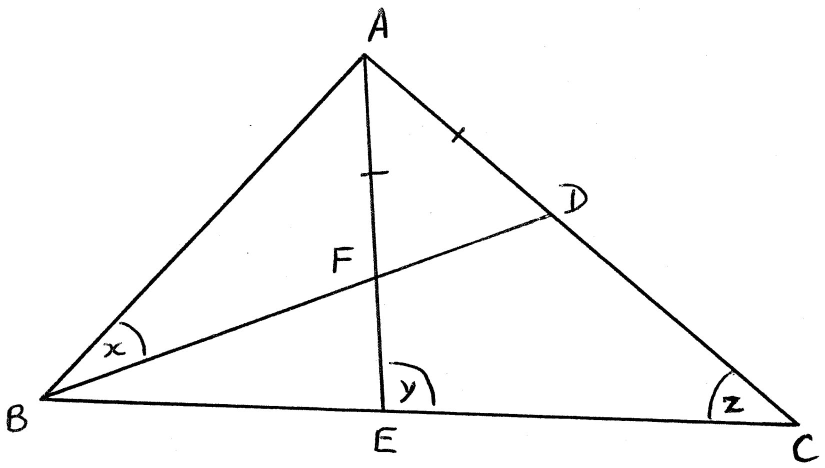
(iii) Hence prove that ΔBPC is isosceles. 2

Question 3 – (14 marks) – Start a new page **Marks**

a) (i) If $\theta = \tan^{-1} A + \tan^{-1} B$ show that $\tan\theta = \dfrac{A+B}{1-AB}$ 1

(ii) Hence solve the equation $\tan^{-1} 3x + \tan^{-1} 2x = \dfrac{\pi}{4}$ 4

b)



The diagram shows triangle ABC. The bisector of angle B meets the line AE at F and the line AC at D.

If $\angle ABD = x^\circ$

$\angle AEC = y^\circ$

$\angle ACB = z^\circ$

and $AD = AF$ show that

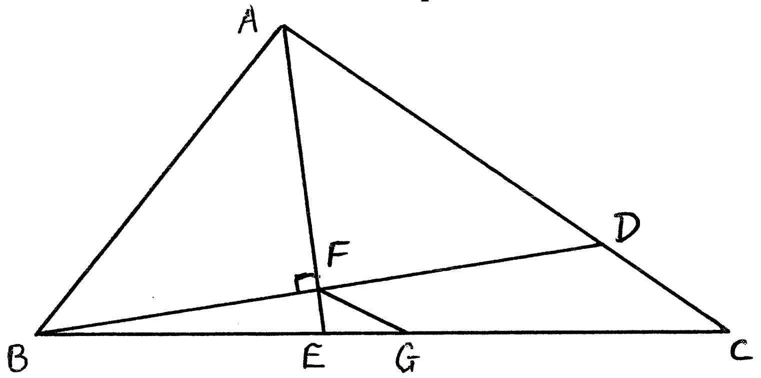
(i) $\angle ADF = \dfrac{1}{2}(y+z)$ 3

(ii) $\angle EAB = y - 2x$ 2

c) Find the equation of the tangent to the curve $y = 3\cos^{-1}\dfrac{x}{2}$ at the point on the curve where $x = 0$ 4

Question 4 – (14 marks) – Start a new page **Marks**

a) In the diagram AE bisects $\angle BAC$, BF is perpendicular to AE and G is the midpoint of BC. BF meets AC at the point D.



(i) Copy this diagram onto your answer sheet and mark in all the given information. 1

(ii) Prove that ΔBAF is congruent to ΔDAF 3

(iii) Explain why $BF = FD$ 1

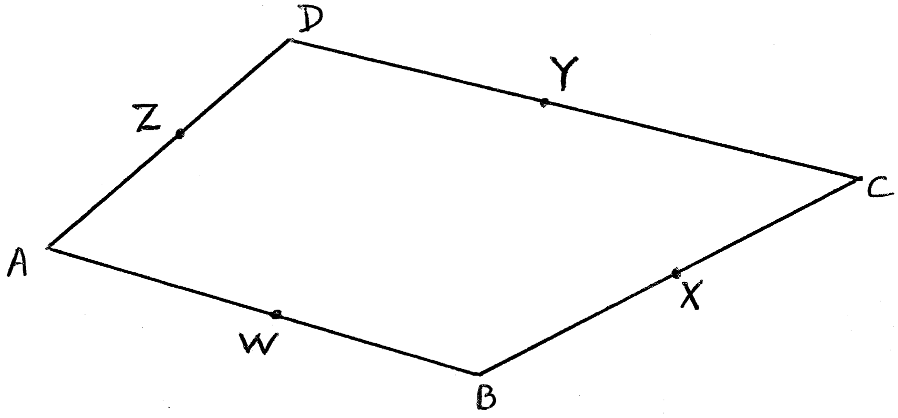
(iv) Hence prove that FG is parallel to DC 4

b) (i) Find $\frac{d}{dx}\left(\sqrt{1-x^2} + x\sin^{-1} x\right)$ 3

(ii) Hence or otherwise evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x \, dx$ correct to 3 significant figures. 2

Question 5 – (14 marks) – Start a new page **Marks**

a)



ABCD is a quadrilateral. If *W* is the midpoint of *AB*, *X* is the midpoint of *BC*, *Y* is the midpoint of *CD* and *Z* is the midpoint of *AD*, prove that the quadrilateral *WXYZ* is a parallelogram.

b) Consider the function $f(x)=\dfrac{8}{4+x^2}$

(i) Show that $f(x)$ is an even function, and the x axis is a horizontal asymptote to the curve $y=f(x)$ 3

(ii) Find the coordinates and nature of the stationary point on the curve $y=f(x)$ 3

(iii) Sketch the graph of the curve showing the above features. 2

(iv) Find the exact area of the region in the first quadrant bounded by the curve $y=f(x)$ and the line $x=2$ 2

End of Paper

Yr 12 Mathematics Ext 1 Assessment Task 3

SOLUTIONS

Question 1

a) $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$u = x \qquad v = \tan^{-1}\frac{x}{2}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{2}{4-x^2}$

$\frac{dy}{dx} = \tan^{-1}\frac{x}{2} + \frac{2x}{4-x^2}$

b) (i) For ΔAXY and ΔABC

$\angle A$ is common

$\angle AXY = \angle ABC$ (corresponding angles on $XY \parallel BC$)

$\therefore \Delta AXY \parallel\!\!| \Delta ABC$ (equiangular)

(ii) $XY : BC = 8 : 14$ sides of similar triangles

$= 4 : 7$

$AY : AC = 4 : 7$ "

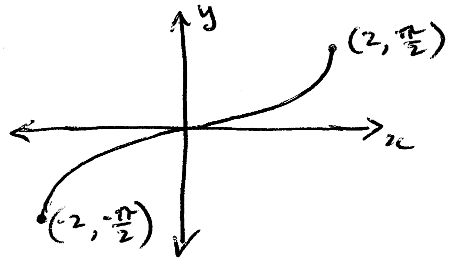
$\frac{AY}{AC} = \frac{4}{7}$

$\frac{AY}{18} = \frac{4}{7}$

$AY = \frac{72}{7}$ cm

c) (i) Domain $-2 \le x \le 2$ range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(ii)



Question 2

a) Let $\angle AED = x$

$\therefore \angle AEB = x$

Let $\angle CEF = y$

$\therefore \angle BEF = y$

$x + x + y + y = 180°$ angles on a straight line

$2x + 2y = 180$

$x + y = 90$

$\angle AEF = x + y$

$= 90°$

b) $\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \left[\frac{1}{\sqrt{3}}\sin^{-1}\frac{\sqrt{3}x}{2}\right]$

$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - 0\right)$

$= \frac{\pi}{3\sqrt{3}}$

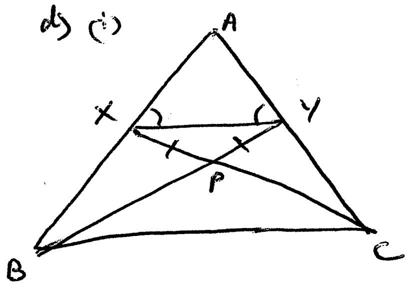
c) $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$\sin^{-1}\left(-\frac{1}{2}\right) = \frac{-\pi}{6}$

$\cos^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} - \frac{-\pi}{6}$

$= \frac{5\pi}{6}$

d) (i)



(ii) For ΔABY and ΔACX

$\angle PXY = \angle PYX$ angles opposite equal sides

$\therefore \angle AXC = \angle AYB$ sums of equal angles

$\angle A$ is common

$AX = AY$ sides opposite equal angles

$\therefore \Delta ABY \equiv \Delta ACX$ (AAS)

(iii) $BY = CX$ (corresponding sides of congruent Δ's)

$BY - PY = CX - PX$

$BP = CP$

$\therefore \Delta BPC$ is an isosceles Δ

Question 3

a)(i) $\tan\theta = \tan(\tan^{-1}A + \tan^{-1}B)$

$$= \frac{\tan(\tan^{-1}A) + \tan(\tan^{-1}B)}{1 - \tan(\tan^{-1}A)\tan(\tan^{-1}B)}$$

$$= \frac{A+B}{1-AB}$$

(ii) $A = 3x \quad B = 2x$

$$\tan^{-1}3x + \tan^{-1}2x = \frac{3x+2x}{1-3x\times 2x}$$

$$= \frac{5x}{1-6x^2}$$

$$\frac{5x}{1-6x^2} = \tan\frac{\pi}{4}$$

$$\frac{5x}{1-6x^2} = 1$$

$$5x = 1-6x^2$$

$$6x^2+5x-1=0$$

$$6x^2+6x-x-1=0$$

$$6x(x+1)-(x+1)=0$$

$$(x+1)(6x-1)=0$$

$$x=-1 \text{ or } \frac{1}{6}$$

b) (i) $\angle AFD = \angle ADF$ angles opposite equal sides

$$\angle EAC = 180-(y+z)$$

$$\angle AFD + \angle ADF = y+z$$

$$\therefore \angle ADF = \frac{y+z}{2}$$

(ii) $\angle ABE + \angle BAE = \angle AEC$ (exterior angle of a triangle)

$$2x + \angle EAB = y$$

$$\angle EAB = y-2x$$

c) $$\frac{dy}{dx} = \frac{-3}{\sqrt{4-x^2}}$$

$$= \frac{-3}{2} \text{ when } x=0$$

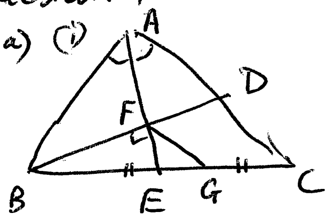
$$\left(0, \frac{3\pi}{2}\right)$$

$$y - \frac{3\pi}{2} = \frac{-3}{2}x$$

$$3x+2y-3\pi=0$$

Question 4

a) (i)



(ii) For Δs ABF and ΔDAF

$\angle BAF = \angle DAF$ given

$\angle BFA = \angle DFA$ given as perpendicular

AF is common

$\therefore \Delta ABF \equiv \Delta ADF$ (AAS)

(iii) $BF = FD$ corresponding sides of congruent triangles

(iv) For ΔBEF and ΔBCD

$BF = FD$ above

$BG = GC$ given

$\angle B$ is common

$\therefore \Delta BEF \,|||\, \Delta BCD$ two sides in proportion and the included angle equal

$\therefore \angle BGF = \angle BCD$ corresponding angles of similar triangles

$\therefore FG \parallel DC$ (corresponding angles are equal on parallel lines)

b)(i) $$\frac{d}{dx}\left(\sqrt{1-x^2} + x\sin^{-1}x\right)$$

$$= \frac{-1}{2}2x\left(\frac{1}{\sqrt{1-x^2}}\right) + \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$$

$$= \frac{-x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$$

$$= \sin^{-1}x$$

(ii) $$\int_0^{\frac{1}{2}} \sin^{-1}x\, dx$$

$$= \left[\sqrt{(1-x^2)} + x\sin^{-1}x\right]_0^{\frac{1}{2}}$$

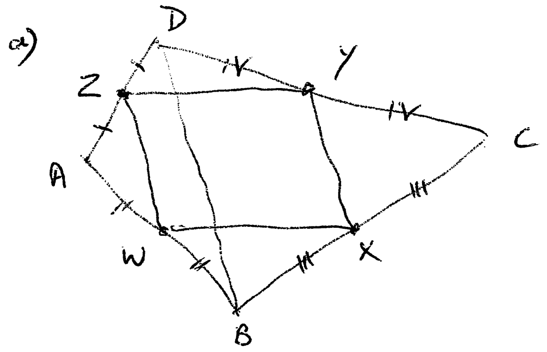
$$= \left[\sqrt{1-\tfrac{1}{4}} + \tfrac{1}{2}\sin^{-1}\tfrac{1}{2} - 1 - 0\right]$$

$$= \frac{\sqrt{3}}{2} + \frac{\pi}{12} - 1$$

$$= 0.128$$

Question 5

a)



join BD

For $\triangle AWZ$ and ABD

$AZ = ZD$

$AW = WB$

$\angle A$ is common

$\therefore \triangle AWZ \; ||| \; \triangle ABD$

two sides in proportion and the included angles equal

$\therefore WZ \parallel BD$ equal intercepts

For $\triangle CXY$ and $\triangle CBD$

$CX = XB$

$CY = YD$

$\angle C$ is common

$\therefore \triangle CXY \; ||| \; \triangle CBD$

two sides in proportion and the included angles equal

$\therefore XY \parallel BD$ equal intercepts

$\therefore WZ \parallel XY$

join AC and use similar proof of

$WX \parallel ZY$

$\therefore WXYZ$ is a parallelogram

b) (i) $f(-x) = \dfrac{8}{4+(-x)^2}$

$= \dfrac{8}{4+x^2}$

$= f(x)$

$\therefore$ it is an even function

$f(x) = \dfrac{\frac{8}{x^2}}{\frac{4}{x^2}+1}$

$\lim_{x\to\infty} \dfrac{\frac{8}{x^2}}{\frac{4}{x^2}+1} = \dfrac{0}{0+1}$

$= 0$

$\therefore$ horizontal asymptote at $f(x) = 0$

(ii) $f'(x) = \dfrac{-16x}{(4+x^2)^2}$

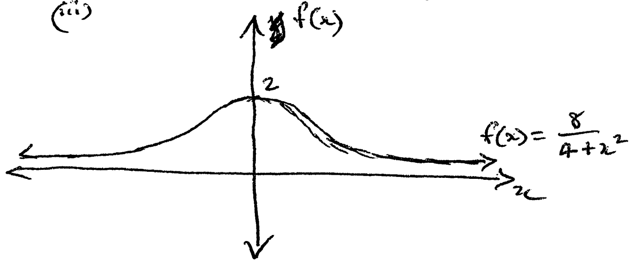
$f'(x) = 0$ when $x = 0$

$f(x) = 2$

x	0^-	0	0^+
	+ve	0	-ve

$\therefore$ a maximum turning pt at $(0,2)$

(iii)



(iv) $\int_0^2 \dfrac{8}{4+x^2}\,dx = \left[4\tan^{-1}\dfrac{x}{2}\right]_0^2$

$= 4\tan^{-1}1 - 4\tan^{-1}0$

$= \pi - 0$

$= \pi$