St George Girls High School

Year 12

Assessment Task 3

2008

Mathematics Extension 1

General Instructions

- Time allowed – 75 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question on a new page.

Total marks – 66

- Attempt Questions 1 – 6
- All questions are of equal value

Question	Mark
Question 1	/11
Question 2	/11
Question 3	/11
Question 4	/11
Question 5	/11
Question 6	/11
Total	/66

Question 1 (11 marks)

Marks

a) Evaluate $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ (2)

b) The polynomial $P(x) = 2x^3 - 5x^2 - 3x + 1$ has zeroes α, β, γ. Evaluate:

i. $\alpha + \beta + \gamma, \quad \alpha\beta + \alpha\gamma + \beta\gamma, \quad \alpha\beta\gamma$ (3)

ii. $3\alpha + 3\beta + 3\gamma - 4\alpha\beta\gamma$ (2)

iii. $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ (1)

c) Find the equation of the tangent to the curve $y = \tan^{-1}(ax+b)$ at the point where the curve crosses the x-axis. (3)

Question 2 (11 marks)

a) Differentiate $x\sin^{-1} 2x$ (2)

b) The cubic polynomial equation $x^3 = ax^2 + bx + c$ has three real roots, two of which are opposites. Prove that:

i. one of the roots is a (1)

ii. the other roots are $\sqrt{b}, \ -\sqrt{b}$ (2)

iii. $ab + c = 0$ (2)

c) Sketch $y = 3\cos^{-1}\frac{x}{2}$. Give the domain and range. (4)

Question 3 (11 marks)

Marks

a) i. Express $\sin x - \sqrt{3}\cos x$ in the form $A\sin(x-\alpha)$ with $A > 0$ and $0 < \alpha < \frac{\pi}{2}$. (4)

ii. Find the general solution to $\sin x - \sqrt{3}\cos x = \frac{2}{\sqrt{2}}$ (2)

b) Solve the equation $5\cos^2 x + \sin^2 x = 3\sin 2x$ for $0^\circ \le x \le 360^\circ$ (3)

c) Explain why the graph of a cubic polynomial with three distinct zeroes must have two turning points. (2)

Question 4 (11 marks)

a) Find the value of $\sin^{-1}(\cos\frac{2\pi}{3})$ (2)

b) i. Show that the equation $3\sin x - 2\cos x = 2$ can be written as $3t - 2 = 0$ where $t = \tan\frac{x}{2}$. (3)

ii. Hence solve the equation for $0^\circ \le x \le 360^\circ$ giving solutions correct to the nearest minute. (3)

c) The polynomial $y = 2x^3 - x^2 + ax + b$ has a remainder of 16 when divided by $x - 1$ and a remainder of -17 when divided by $x + 2$. Find a and b. (3)

Question 5 (11 marks) **Marks**

a) Show that if $x-2$ is a factor of x^3+ax^2+bx+c it is also a factor of $ax^3+bx^2+cx+16$ (4)

b) i. Factorise the polynomial $P(x)=x^3-6x^2-9x+14$ completely. (4)

ii. Without using calculus, sketch the graph of $P(x)$, showing all intercepts with the axes. (3)

c) Evaluate $\sin[(2\tan^{-1}(1))]$ (2)

Question 6 (11 marks)

a) If $y=\tan^{-1}x$

i. Prove that $\sec^2 y=1+x^2$ (3)

ii. Hence or otherwise prove that $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$ (2)

b) i. If $\theta=\tan^{-1}A+\tan^{-1}B$ show that $\tan\theta=\frac{A+B}{1-AB}$ (3)

ii. Hence solve $\tan^{-1}3x+\tan^{-1}2x=\frac{\pi}{4}$ (3)

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

QUESTION 1

a) $\int_0^3 \frac{dx}{\sqrt{3^2 - x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^3$

$= \sin^{-1} 1 - \sin^{-1} 0$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$

b) (i) $a = 2$, $b = -5$, $c = -3$, $d = 1$

$\alpha + \beta + \gamma = \frac{-b}{a}$ $\qquad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ $\qquad \alpha\beta\gamma = \frac{-d}{a}$

$= \frac{5}{2}$ $\qquad = \frac{-3}{2}$ $\qquad = \frac{-1}{2}$

(ii) $3(\alpha + \beta + \gamma) - 4\alpha\beta\gamma$

$= 3 \times \frac{5}{2} - 4 \times \left(-\frac{1}{2}\right)$

$= \frac{15}{2} + 2$

$= \frac{19}{2}$

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= -\frac{3}{2} \div -\frac{1}{2}$

$= 3$

c) the curve crosses the x axis when $y = 0$

$\tan^{-1}(ax + b) = 0$

$(ax + b) = 0$

$ax = -b$

$x = \frac{-b}{a}$

$$y' = \frac{a}{1+(ax+b)^2}$$

when $x = -\frac{b}{a}$

$$y' = \frac{a}{1+\left(a\left(-\frac{b}{a}\right)+b\right)^2}$$

$$= a$$

Equation of the tangent:

$$y - 0 = a\left(x + \frac{b}{a}\right)$$

$$y = ax + b$$

Question 2

a) $$\frac{d}{dx}(x \sin^{-1} 2x) = \sin^{-1} 2x + \frac{2}{\sqrt{1-(2x)^2}}$$

$$= \sin^{-1} 2x + \frac{2}{\sqrt{1-4x^2}}$$

b) Let the roots be $\alpha, \beta, -\beta$

(i) $$\alpha + \beta + (-\beta) = \frac{a}{1}$$

$$\alpha = a$$

(ii) $$\alpha\beta + \alpha(-\beta) - \beta^2 = -b$$

$$\alpha\beta - \alpha\beta - \beta^2 = -b$$

$$-\beta^2 = -b$$

$$\beta^2 = b$$

$$\beta = \pm\sqrt{b}$$

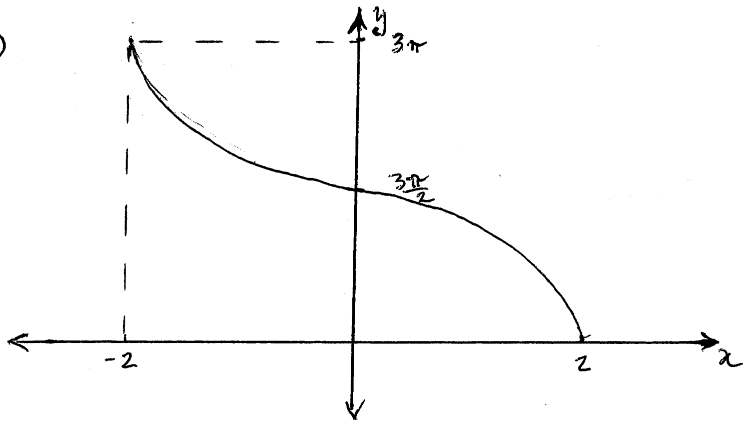
(iii) $$\alpha\beta(-\beta) = c$$

$$-\alpha\beta^2 = c \quad \text{since } \alpha = a, \ \beta^2 = b$$

$$-ab = c$$

$$ab + c = 0$$

c)



Domain $-2 \leq x \leq 2$

Range $0 \leq y \leq 3\pi$

Question 3

a) (i) $\sin x - \sqrt{3}\cos x = A\sin(x-\alpha)$

$A = \sqrt{a^2+b^2}$, where $a = 1$, $b = \sqrt{3}$

$= \sqrt{1+3}$

$= \sqrt{4}$ 1

$= 2$

$\tan\alpha = \frac{\sqrt{3}}{1}$ 1

$\alpha = \frac{\pi}{3}$

$\sin x - \sqrt{3}\cos x = 2\sin\left(x - \frac{\pi}{3}\right)$ 1

(ii) $2\sin\left(x-\frac{\pi}{3}\right) = \frac{2}{\sqrt{2}}$

$\sin\left(x-\frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$ 1

$x - \frac{\pi}{3} = \frac{\pi}{4} + 2n\pi \,,\; \frac{3\pi}{4} + 2n\pi$ 1

where n is an integer

$x = \frac{7\pi}{12} + 2n\pi \,,\; \frac{13\pi}{12} + 2n\pi$ 1

$n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{3} \Rightarrow \frac{\pi}{3}(3n+1) + \ldots$

b) $5\cos^2 x + \sin^2 x = 3\times 2\sin x\cos x$

$= 6\sin x\cos x$

1

$5\cos^2 x - 6\sin x\cos x + \sin^2 x = 0$ $\quad 0° \leq x \leq 360°$

$(5\cos x - \sin x)(\cos x - \sin x) = 0$

either $5\cos x - \sin x = 0$ or $\cos x - \sin x = 0$

$5\cos x = \sin x$ $\quad\quad \cos x = \sin x$

$5 = \tan x$ 1 $\quad\quad 1 = \tan x$ 1

$x = 78°41', 258°41'$ $\quad\quad x = 45°, 225°$

c) If a cubic polynomial has 3 distinct zeroes then it must cross the x axis three times (each zero cannot have a multiplicity greater than 1. For a curve to cross the x axis 3 times it must change direction twice which means it must have two turning points. $-\frac{1}{2}$

$13\frac{1}{2}$ x

Question 4

a) Let $x = \cos\frac{2\pi}{3}$

$= -\frac{1}{2}$

$\sin^{-1}\left(-\frac{1}{2}\right) = \frac{-\pi}{6}$

b) (i) $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, where $t = \tan\frac{x}{2}$

$$3\left(\frac{2t}{1+t^2}\right) - 2\left(\frac{1-t^2}{1+t^2}\right) = 2$$

$$\frac{6t-2+2t^2}{1+t^2} = 2$$

$$\frac{6t-2+2t^2-2(1+t^2)}{1+t^2} = 0$$

$$\frac{6t-2+2t^2-2-2t^2}{1+t^2} = 0$$

$$\frac{6t-4}{1+t^2} = 0$$

$$\therefore 6t-4 = 0$$

$$3t-2 = 0$$

(ii) $t = \frac{2}{3}$

$\tan\frac{x}{2} = \frac{2}{3}$ $\qquad 0° \leq x \leq 360°$

$\qquad 0° \leq \frac{x}{2} \leq 180°$

$\frac{x}{2} = 33°41'$

$x = 67°22'$ $\qquad$ NOTE: CHECK $x = 180°$!

c) $P(x) = 2x^3 - x^2 + ax + b$

$P(1) = 16$

$2-1+a+b = 16$

$a+b = 15$ ——— ①

$P(-2) = -17$

$-16-4-2a+b = -17$

$-2a+b = 3$ ——— ②

① − ②

$3a = 12$

$a = 4$

$b = 11$

Question 5

a) Let $P(x) = x^3 + ax^2 + bx + c$

if $x-2$ is a factor of $P(x)$ then $P(2) = 0$

$$P(2) = 8 + 4a + 2b + c$$

$$8 + 4a + 2b + c = 0$$

$$4a + 2b + c = -8$$

Let $Q(x) = ax^3 + bx^2 + cx + 16$

if $x-2$ is a factor of $Q(x)$ then $Q(2) = 0$

$$Q(2) = 8a + 4b + 2c + 16$$
$$= 2(4a + 2b + c) + 16$$
$$= 2 \times (-8) + 16$$
$$= 0$$

$\therefore$ $x-2$ is a factor of $ax^3 + bx^2 + cx + 16$ if it is a factor of $x^3 + ax^2 + bx + c$

b) (i) $P(x) = x^3 - 6x^2 - 9x + 14$

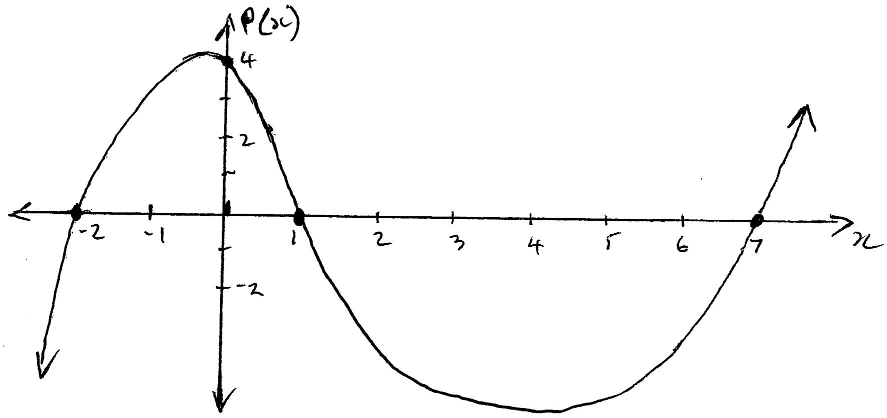
$$P(1) = 1 - 6 - 9 + 14$$
$$= 0$$

$\therefore$ $x - 1$ is a factor of $P(x)$

$$\begin{array}{r} x^2 - 5x - 14 \\ x-1 \enclose{longdiv}{x^3 - 6x^2 - 9x + 14} \\ \underline{x^3 - x^2} \qquad\qquad \\ -5x^2 - 9x \qquad \\ \underline{-5x^2 + 5x} \qquad \\ -14x + 14 \\ \underline{-14x + 14} \\ 0 \end{array}$$

$$P(x) = (x-1)(x^2 - 5x - 14)$$
$$= (x-1)(x-7)(x+2)$$

(ii)



c) $\tan^{-1}(1) = \frac{\pi}{4}$

$\sin \frac{2\pi}{4}$

$= \sin \frac{\pi}{2}$

$= 1$

Question 6

a) (i) $\tan y = x$

$\tan^2 y = x^2$

$\tan^2 y + 1 = 1 + x^2$

$\sec^2 y = 1 + x^2$

(ii) Let $y = \tan^{-1} x$

$x = \tan y$

$\frac{dx}{dy} = \sec^2 y$

$= 1 + x^2$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

b) (i) $\tan \theta = \tan(\tan^{-1} A + \tan^{-1} B)$

$= \frac{\tan(\tan^{-1} A) + \tan(\tan^{-1} B)}{1 - \tan(\tan^{-1} A)\tan(\tan^{-1} B)}$

$= \frac{A + B}{1 - AB}$

(ii) $\tan(\tan^{-1} 3x + \tan^{-1} 2x) = \tan \frac{\pi}{4}$

$\frac{3x + 2x}{1 - 3x \times 2x} = 1$

$\frac{5x}{1 - 6x^2} = 1$

$5x = 1 - 6x^2$

$6x^2 + 5x - 1 = 0$

$(6x - 1)(x + 1) = 0$

$x = \frac{1}{6}$ or $x = -1$

NOTE: CHECK both these solutions!