SECTION A

QUESTION 1 **Marks**

(a) Differentiate with respect to x:

(i) e^{-2x} **4**

(ii) xe^{x^2}

(iii) $\sin^{-1} 3x$

(iv) $\log_e \cos^2 x$

(b) Find a primitive of **3**

(i) $\dfrac{1}{2x+3}$

(ii) $\dfrac{x+2}{x^2+4}$

(c) Evaluate **3**

$$\int_0^{\frac{\pi}{2}} (\cos x - \sin x)^2 \, dx$$

QUESTION 2 **Marks**

(a) A particle, moving in a straight line, starts from the origin with velocity 15 m/s. Its acceleration is given by $a = 7 - 4t$ where t is the time in seconds. **3**

(i) When does the particle come to rest ?

(ii) How far has it travelled at that time ?

(b) A circular oil slick lies on the surface of a body of calm water. Its area is increasing at a rate of 500 m^2 per hour. At what rate is its circumference increasing when the radius of the slick is 12.5 m ? **2**

(c) (i) Show that one root of the equation **3**

$$e^x - 2\cos x - 1 = 0$$

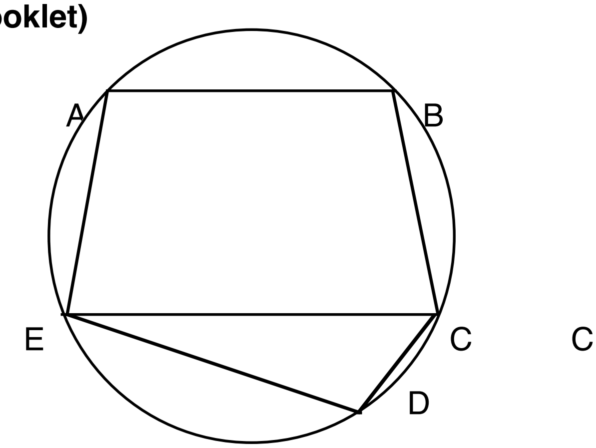
lies between 0.5 and 1

(ii) Use Newton's Method once to find a better approximation than 0.75 for the root of the above equation.

(d) Find the equation of the normal to the curve $y = \log_e x^2$ at the point where $x = e$ **2**

SECTION B

<u>QUESTION 3</u> (Start a new booklet) **Marks**



(a) In the diagram above the chords AB and EC are parallel. **2**

Copy the diagram onto your booklet.

Show that $\angle$ ADE = $\angle$ BDC

(b) For the curve **4**

$y = (x - 2)^2 - 1$

(i) What is the largest positive domain for $f(x)$ to have an inverse function?

(ii) Find this inverse function, stating its Domain and Range.

(iii) Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes.

(c) A particle, moving in a positive direction in a straight line, has acceleration **4**

$\ddot{x} = -\frac{1}{2}e^{-x}\,cms^{-2}$. Initially the velocity (v) =1 cms^{-1} and the displacement (x)=0 cm.

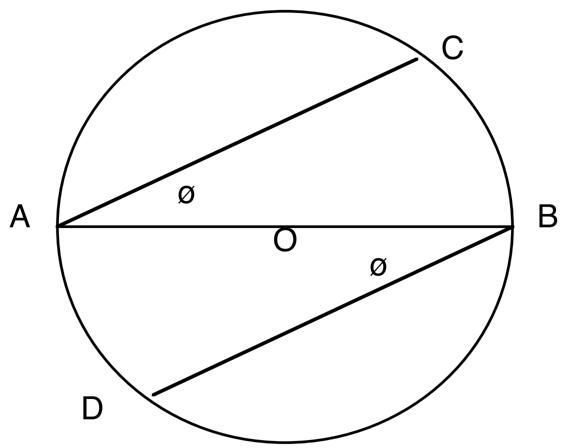
(i) Find the velocity of the particle when it is 2 cm from the origin.

(ii) Find an expression for the particle's displacement with respect to time.

QUESTION 4 **Marks**

(a) (i) Show that $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$ **1**

(b)



In the diagram above AB is a diameter of a circle with centre O and radius r . **6**
The chords AC and DB each make an angle ø with AB. The area between the chords is $\frac{3}{4}$ of the area of the circle. Copy the diagram into your booklet.

(i) Show that 2ø + sin 2ø = $\frac{3\pi}{4}$

(ii) Join CO and OD and show that CD is a diameter of the circle.

(iii) The point C moves along the circumference at a constant rate, moving through one quadrant in 15 minutes. If A is the total area of the two triangles ACO and DBO, find, in terms of r, the rate at which A is changing when ø = $\frac{\pi}{6}$

(iv) Find the maximum value of A and the value of ø at that time.

(c) (i) Sketch the curve $y = 3\sin 2x$ $\quad 0 \leq x \leq \pi$ **3**

(ii) The area between the curve $y = 3\sin 2x$, $x = 0$ and $x = \pi$ is rotated about the x axis. Find, in terms of π, the volume of the solid generated.

SECTION C

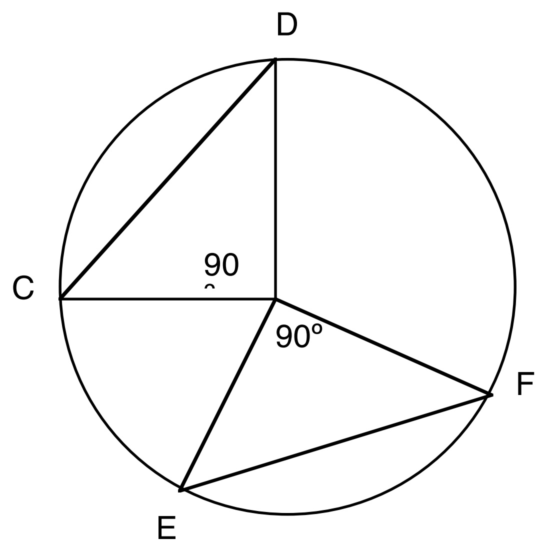
QUESTION 5 **(Start a new booklet)** **Marks**

(a) If $\tan^{-1} x = \frac{\pi}{6}$, find the value of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ **1**

(b) Find $\lim\limits_{x \to 0} (\text{cosec x - cot x})$ **2**

(c) Evaluate and simplify: $\frac{d}{dx}\left(\sin^{-1}\frac{2x}{3}\right)$ **2**

(d)



The chords CD and EF each subtend an angle of 90° at the centre of the circle. Show that the chords DE and CF meet at right angles. **2**

(e) Use the substitution $u = \tan x$ to evaluate **3**

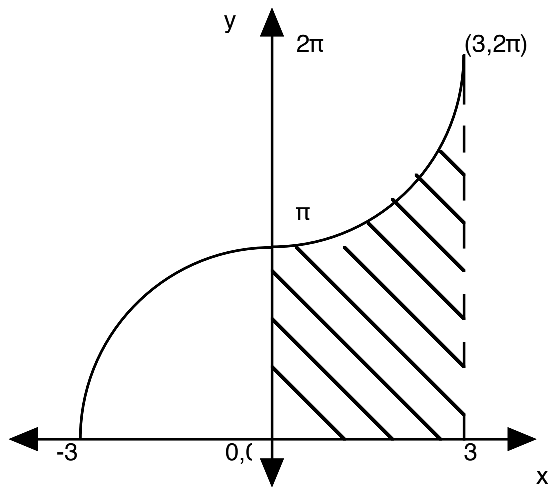
$$\int_0^{\frac{\pi}{4}} \frac{dx}{9\cos^2 x + 25\sin^2 x}$$

QUESTION 6 **Marks**

(a) Show by Mathematical Induction that **5**

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\cdots+\frac{1}{4n^2-1}=\frac{n}{2n+1} \qquad \text{for all } n \geq 1$$

(b)



The graph above is of the function $y = A\cos^{-1} Bx$ where A and B are constants. **5**

(i) Find the values of A and B.

(ii) Find the exact area of the shaded region.

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2002
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 3

Mathematics Extension 1

Sample Solutions

Question 1

(a) (i) $\frac{d(e^{-2x})}{dx} = -2e^{-2x}$

(ii) $\frac{d(xe^{x^2})}{dx} = e^{x^2} + x(2xe^{x^2}) = e^{x^2}(1+2x^2)$

(iii) $\frac{d(\sin^{-1}3x)}{dx} = \frac{3}{\sqrt{1-9x^2}}$

(iv) $\ln(\cos^2 x) = 2\ln\cos x$

$\therefore \frac{d(2\ln\cos x)}{dx} = 2\cdot\frac{(-\sin x)}{\cos x} = -2\tan x$

(b) (i) $\int \frac{1}{2x+3}\,dx = \frac{1}{2}\int \frac{2\,dx}{2x+3} = \frac{1}{2}\ln|2x+3| + C$

(ii) $\int \frac{x+2}{x^2+4}\,dx = \int \frac{x}{x^2+4}\,dx + \int \frac{2}{x^2+4}\,dx$

$= \frac{1}{2}\int \frac{2x}{x^2+4}\,dx + \tan^{-1}\left(\frac{x}{2}\right) + C$

$= \frac{1}{2}\ln(x^2+4) + \tan^{-1}\left(\frac{x}{2}\right) + C$

(c) $\int_0^{\pi/2} (\cos x - \sin x)^2\,dx = \int_0^{\pi/2} (\cos^2 x + \sin^2 x - 2\sin x\cos x)\,dx$

$= \int_0^{\pi/2} (1 - \sin 2x)\,dx$

$= \left[x + \frac{1}{2}\cos 2x\right]_0^{\pi/2}$

$= \left(\frac{\pi}{2} + \frac{1}{2}\times -1\right) - \left(0 + \frac{1}{2}\right)$

$= \frac{\pi}{2} - 1$

(2) (a) $t=0, x=0, v=15$

$a = 7-4t$

(i) $v = \int a\,dt = 7t - 2t^2 + c$

$\therefore v = 7t - 2t^2 + 15$

$v=0 \Rightarrow 2t^2 - 7t - 15 = 0$

$(2t+3)(t-5) = 0$

$\therefore \underline{t=5}$

(ii) distance $= \int_0^5 (7t - 2t^2 + 15)\,dt$

$= \frac{7}{2}t^2 - \frac{2}{3}t^3 + 15t \Big]_0^5$

$= \frac{7}{2}\times 25 - \frac{2}{3}\times 125 + 15\times 5 - 0$

$=$

(b) $A = \pi r^2$

$\frac{dA}{dt} = 500 = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 2\pi r \times \frac{dr}{dt}$

$500 = 2\pi \times 12.5 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{500}{25\pi} =$

$\therefore C = 2\pi r$

$\frac{dC}{dt} = 2\pi \frac{dr}{dt} =$

(c) (i) $f(x) = e^x - 2\cos x - 1$, f is a continuous function

$f(0.5) = e^{0.5} - 2\cos 0.5 - 1 \doteqdot -1.11$ [Remember to use RADIANS]

$f(1) = e - 2\cos 1 - 1 \doteqdot 0.64 > 0$

$\therefore f(1) \times f(0.5) < 0$, so with f continuous $f(c) = 0$ with $0.5 < c < 1$

(ii) $f'(x) = e^x + 2\sin x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.75 - \frac{e^{0.75} - 2\cos 0.75 - 1}{e^{0.75} + 2\sin 0.75}$

$\doteqdot 0.85$

(2) (d) $y = \log_e x^2 = 2\log_e x$

$x = e \quad y = 2\ln e = 2$

$y' = \frac{2}{x} \Rightarrow$ at $x = e \quad y' = \frac{2}{e}$

$\therefore m_\perp = -\frac{e}{2}$

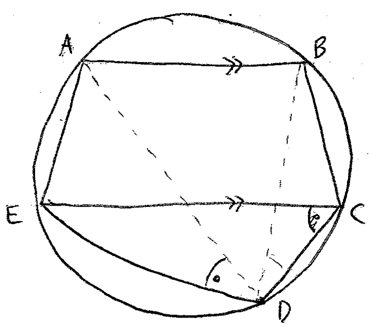
$\therefore y - 2 = -\frac{e}{2}(x - e)$

$\therefore 2y - 4 = -ex + e^2$

$\therefore ex + 2y + e^2 - 4 = 0$

(3)

(a)



ABCE is an isosceles trapezium

$\angle E + \angle B = 180^\circ$ (cyclic quad)

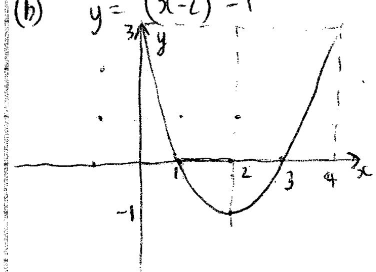
$\angle C + \angle \quad = 180^\circ$ (co-interior)

$\therefore \angle E = \angle C$

$\therefore AE = BC$

$\therefore \angle BDC = \angle AE$ (equal chords subtend equal angles at the circumference)

(b) $y = (x-2)^2 - 1$

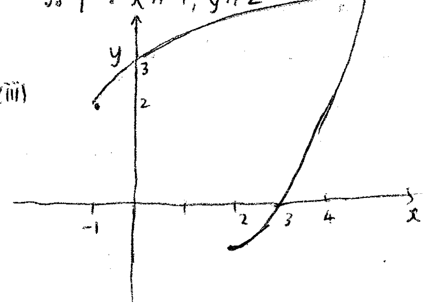


(i) $x \geq 2$

(ii) $f: x \geq 2, \ y \geq -1$

$\therefore f^{-1}: x \geq -1, \ y \geq 2$

(iii)



3 (c) $t=0,\ v \geq 0$

$x=0$

$a = -\frac{1}{2}e^{-x}$ $\qquad \frac{d(\frac{1}{2}v^2)}{dx} = -\frac{1}{2}e^{-x}$

$$\tfrac{1}{2}v^2 = \tfrac{1}{2}e^{-x} + C$$

$$v^2 = e^{-x} + C_1$$

$$\therefore C_1 = 0 \Rightarrow v^2 = e^{-x}$$

$(v \neq 0)$ $\quad \therefore v = e^{-x/2}$ $\quad$ (positive square root)

(i) $x=2 \Rightarrow v = e^{-1} = \frac{1}{e}$ cm/s

(ii) $v = e^{-x/2}$

$\therefore \frac{dx}{dt} = e^{-x/2} \Rightarrow \frac{dt}{dx} = e^{x/2}$

$$t = \tfrac{1}{2}e^{x/2} + C_2$$

$$0 = \tfrac{1}{2} + C_2$$

$$\therefore t = \tfrac{1}{2}(e^{x/2} - 1)$$

$$\therefore e^{x/2} = 2t+1$$

$$\therefore x/2 = \ln(2t+1)$$

$$\therefore x = 2\ln(2t+1)$$

(4) (a) For $x>0$ $\quad \tan^{-1}(x) + \tan^{-1}(\frac{1}{x}) = \frac{\pi}{2}$

Let $f(x) = \tan^{-1}x + \tan^{-1}\frac{1}{x}$

$\therefore f'(x) = \frac{1}{1+x^2} + \left(\frac{-1}{x^2}\right)\frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$, $x>0$

$\therefore f(x) = k$, constant

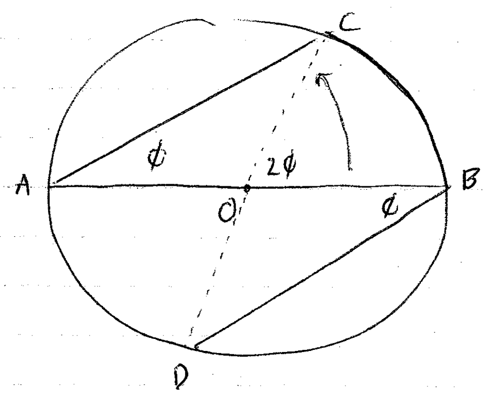
$f(1) = k$

[NB if $x<0$ $\quad \tan^{-1}x + \tan^{-1}\frac{1}{x} = -\frac{\pi}{2}$]

$\therefore \quad = \tan^{-1}(1) + \tan^{-1}(1) = 2\times\frac{\pi}{4} = \frac{\pi}{2}$

$\therefore \tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$, $x>0$

4 (b)



AC ∥ DB

$B\hat{O}C = 2\phi$ (angle at centre)

sector $BOC = \frac{1}{2}r^2 \cdot 2\phi$
$= r^2\phi$

$\Delta AOC = \frac{1}{2}r^2 \sin(\pi - 2\phi)$
$= \frac{1}{2}r^2 \sin 2\phi$

$$2\left(r^2\phi + \frac{1}{2}r^2\sin 2\phi\right) = \frac{3\pi}{4}r^2$$

$\therefore 2\phi + \sin 2\phi = \frac{3\pi}{4}$ QED

(ii) $\angle AOD = 2\phi$ (angle at the centre)

$\therefore \angle DOB = \pi - 2\phi$ (straight line AB)

$\therefore \angle COB + \angle DOB = \pi$

$\therefore$ C, O, D are collinear

$\therefore$ CD is a diameter.

1 quadrant in 15 mins

(iii) $A = 2 \times \frac{1}{2}r^2 \sin(\pi - 2\phi)$
$= r^2 \sin 2\phi$

$\frac{dA}{dt} = r^2 \cos 2\phi \times 2\frac{d\phi}{dt}$ (chain rule)

$= 2r^2 \cos 2\phi \frac{d\phi}{dt}$

$\angle COB = 2\phi$

$\therefore 2\frac{d\phi}{dt} = \pi/2 / 15$ rads/min

$\therefore \frac{d\phi}{dt} = \pi/60$ rads/min

$r = \frac{\pi}{6} \Rightarrow \frac{dA}{dt} = 2r^2 \cos\frac{\pi}{3} \cdot \frac{\pi}{60}$

$= \frac{r^2\pi}{60}$ units/min

(iv) $\frac{dA}{d\phi} = 2r^2 \cos 2\phi$. So $\frac{dA}{d\phi} = 0 \Rightarrow \cos 2\phi = 0$

$2\phi = \frac{\pi}{2}, \frac{3\pi}{2}$

$\phi = \frac{\pi}{4}, \frac{3\pi}{4}$

$\frac{d^2A}{d\phi^2} = -4r^2 \sin 2\phi$

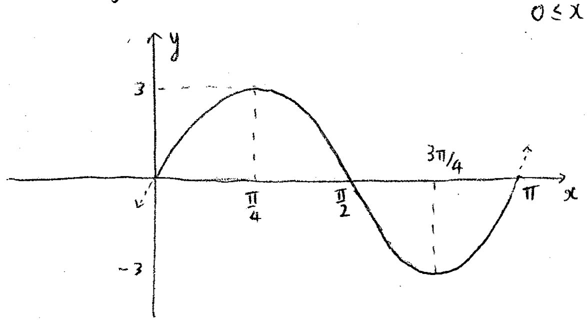
$\phi = \frac{\pi}{4}$ $\frac{d^2A}{d\phi^2} < 0$ $\therefore$ max

(4)(b) (iv) A is max at $\theta = \frac{\pi}{4}$

$$A = r^2 \sin\frac{\pi}{2}$$
$$= r^2 \quad \text{unit}^2$$

(c) (i) $y = 3\sin 2x$

$0 \le x \le \pi$



(ii) $V = \pi \int_0^{\pi} 9\sin^2 2x \, dx$

$$= \frac{9\pi}{2}\int_0^{\pi} 2\sin^2 2x \, dx = \frac{9\pi}{2}\int_0^{\pi}(1-\cos 4x)\,dx$$

$$= \frac{9\pi}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi}$$

$$= \frac{9\pi}{2}\left[(\pi - 0) - (0)\right] = \frac{9\pi^2}{2} \quad \text{unit}^3$$

(5) (a) $\tan^{-1}x = \frac{\pi}{6} \Rightarrow x = \tan\pi/6 = 1/\sqrt{3}$

$$\therefore \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2/\sqrt{3}}{1-\frac{1}{3}}\right) = \tan^{-1}\left(\frac{2/\sqrt{3}}{2/3}\right) = \tan^{-1}(\sqrt{3})$$
$$= \pi/3$$

(b) $\lim_{x\to 0}(\operatorname{cosec}x - \cot x)$

$\operatorname{cosec}x - \cot x$

$$= \operatorname{cosec}x - \cot x \times \frac{\operatorname{cosec}x + \cot x}{\operatorname{cosec}x + \cot x}$$

$$= \frac{\operatorname{cosec}^2 x - \cot^2 x}{\operatorname{cosec}x + \cot x} = \frac{1}{\operatorname{cosec}x + \cot x}$$

$$= \frac{1}{\operatorname{cosec}x + \frac{1}{\tan x}}$$

$$= \frac{\tan x}{\tan x \operatorname{cosec}x + 1}$$

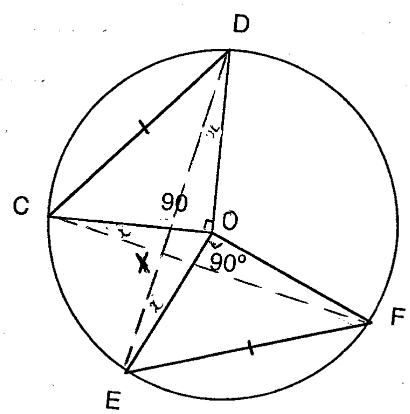
$$\therefore \lim_{x\to 0}(\operatorname{cosec}x - \cot x)$$
$$= \lim_{x\to 0}\frac{\tan x}{\tan x \operatorname{cosec}x + 1}$$
$$= \frac{0}{0+1}$$
$$= 0$$

(5) (c) $\dfrac{d\left(\sin^{-1}\left(\frac{2x}{3}\right)\right)}{dx} = \dfrac{2}{3} \times \dfrac{1}{\sqrt{1-\frac{4x^2}{9}}}$

$= \dfrac{2}{3} \times \dfrac{3}{\sqrt{9-4x^2}}$

$= \dfrac{2}{\sqrt{9-4x^2}}$

(d)



$\angle OCD = 45°$ (isosceles Δ)

Let $\angle OCF = x$

$\therefore \angle EDO = x$ ($\Delta ODE \equiv \Delta OCF$)

$\therefore \angle XCD = 45 + x$

$\angle CDX = 45 - x$

$\therefore \angle DXC = 90°$

$\therefore CF \perp DE$

(e) $\displaystyle\int_0^{\pi/4} \frac{dx}{9\cos^2 x + 25\sin^2 x}$ $\div \cos^2 x$

$= \displaystyle\int_0^{\pi/4} \frac{\sec^2 x\, dx}{9 + 25\tan^2 x}$ $u = \tan x$,

$\therefore x = 0 \Rightarrow u = 0$

$x = \pi/4 \Rightarrow u = 1$

$= \displaystyle\int_0^1 \frac{du}{9+25u^2}$

$= \dfrac{1}{25}\displaystyle\int_0^1 \frac{du}{\frac{9}{25}+u^2} = \dfrac{1}{25} \times \dfrac{5}{3} \tan^{-1}\left(\dfrac{5u}{3}\right)\Big]_0^1$

$= \dfrac{1}{15} \tan^{-1}\left(\dfrac{5}{3}\right)$

$\doteqdot 0.06869$

(6) (a)

$$\frac{1}{3} + \frac{1}{15} + \dots + \frac{1}{4n^2-1} = \frac{n}{2n+1} \qquad n \geq 1$$

Test $n=1$

$$LHS = \frac{1}{3}$$

$$RHS = \frac{1}{2\times1+1} = \frac{1}{3}$$

$\therefore$ True for $n=1$

Assume true for $n=k$

ie. $$\frac{1}{3} + \frac{1}{15} + \dots + \frac{1}{4k^2-1} = \frac{k}{2k+1} \qquad — (*)$$

NTP true for $n=k+1$

ie. $$\frac{1}{3} + \frac{1}{15} + \dots + \frac{1}{4k^2-1} + \frac{1}{4(k+1)^2-1} = \frac{k+1}{2(k+1)+1} = \frac{k+1}{2k+3}$$

$$LHS = \frac{1}{3} + \frac{1}{15} + \dots + \frac{1}{4k^2-1} + \frac{1}{4(k+1)^2-1}$$

$$= \frac{k}{2k+1} + \frac{1}{4(k+1)^2-1} \qquad \text{from } (*)$$

$$4(k+1)^2-1 = (2(k+1)+1)(2(k+1)-1)$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+3)(2k+1)}$$

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(k+1)(2k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3}$$

$= RHS$

$\therefore$ If true for $n=k \Rightarrow$ true for $n=k+1$

$\therefore$ By the principle of mathematical induction then

$$\frac{1}{3} + \frac{1}{15} + \dots + \frac{1}{4n^2-1} = \frac{n}{2n+1}, \text{ for } n \geq 1$$

(6) (b) $y = A\cos^{-1}Bx$ $\qquad \cos^{-1}(0) = \frac{\pi}{2}$

$\therefore A \times \frac{\pi}{2} = \pi$

$\therefore A = 2$

$\Rightarrow y = 2\cos^{-1}Bx$

$(-3,0)$

$0 = 2\cos^{-1}(-3B)$

$\therefore \cos^{-1}(-3B) = 0$

$-3B = \cos 0 = 1$

$\therefore B = \frac{-1}{3}$

OR

$(3, 2\pi)$

$2\pi = 2\cos^{-1}(3B)$

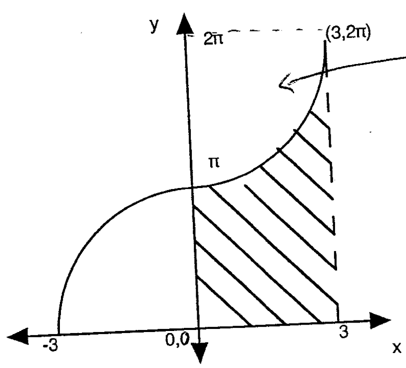
$\pi = \cos^{-1}(3B)$

$3B = \cos\pi$

$B = -\frac{1}{3}$

$\boxed{\cos^{-1}(-x) = \pi - \cos^{-1}(x)}$

$y = 2\cos^{-1}\left(-\frac{x}{3}\right) = 2\pi - 2\cos^{-1}(x/3)$



area

$= \int_{\pi}^{2\pi} -3\cos\left(\frac{x}{2}\right) dx$

$= \left[-6\sin\frac{x}{2}\right]_{\pi}^{2\pi}$

$= -6\left(\sin\pi - \sin\frac{\pi}{2}\right)$

$= 6$

$\therefore$ shaded area $= 6\pi - 6$ sq. units