SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2004

YEAR 12

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 3

Mathematics Extension 1

General Instructions

- Working time – 90 minutes.
- Reading Time – 5 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work

Total Marks - 66

- Attempt *all* questions
- *All* questions are of equal value
- Return your answers in 3 booklets, one for each section. Each booklet must show your student number.

Examiner: *Mr R Dowdell*

Standard Integrals

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left\{x + \sqrt{x^2 - a^2}\right\}, \; |x| > |a|$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left\{x + \sqrt{x^2 + a^2}\right\}$$

NOTE: $\ln x = \log_e x$

Section A:

Question 1: (11 marks) Marks

(a) Evaluate $\displaystyle\int_0^2 \frac{dx}{\sqrt{16-x^2}}$ 2

(b) Evaluate

(i) $\displaystyle\lim_{x \to 0} \frac{\sin 3x}{4x}$

(ii) $\displaystyle\lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$ 3

(c) Use the substitution $u = \ln x$ to find $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$. 2

(d) Differentiate $\log_e(\sin^3 x)$, writing your answer in simplest form. 2

(e) Differentiate with respect to x, $(\tan^{-1} x)^2$. 2

Question 2: (11 marks)

Marks

(a) (i) Write down the domain and range of $y = \sin^{-1}(\sin x)$.

(ii) Draw a neat sketch of $y = \sin^{-1}(\sin x)$. 3

(b) Given that $y = \sin^{-1}(\sqrt{x})$, show that $\dfrac{dy}{dx} = \dfrac{1}{\sin 2y}$. 3

(c) Show that the derivative of $x \tan x - \ln(\sec x)$ is $x \sec^2 x$.

Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$. 3

(d) If $y = 10^x$, find $\dfrac{dy}{dx}$ when $x = 1$. 2

Section B:

Question 3: (11 marks) **START A NEW BOOKLET** Marks

(a) Consider the function $y = 4\sin\left(x + \frac{\pi}{6}\right)$, $\frac{\pi}{3} \le x \le \frac{4\pi}{3}$.

(i) Find the inverse function of y, and write down its domain. 4

(ii) Sketch the inverse function of y.

(b) (i) On the same axes, draw the graphs of $y = \tan^{-1} x$ and $y = \cos^{-1} x$, showing the important features. Mark the point P where the curves intersect. 5

(ii) Show that, if $\tan^{-1} x = \cos^{-1} x$, then $x^4 + x^2 - 1 = 0$. Hence, find the coordinates of P, correct to 2 decimal places.

(c) Show that $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$ 2

Question 4: (11 marks)

Marks

(a) (i) Draw a neat sketch of $y = \cos^{-1} x$. State its domain and range.

(ii) Shade the area bounded by $y = \cos^{-1} x$ and the x and y axes on your diagram. 4

(iii) Calculate the area of the region specified in (ii).

(b) Differentiate $y = \log_e\left(\dfrac{2x}{(x-1)^2}\right)$. Write your answer in simplest form. 2

(c) The rate of change of temperature T^o, of an object is given by the equation $\dfrac{dT}{dt} = k(T-16)$ degrees per minute, k a constant.

(i) Show that the function $T = 16 + Pe^{kt}$, where P is a constant and t the time in minutes, satisfies the equation.

(ii) If initially $T = 0$ and after 10 minutes $T = 12$, find the values of P and k. 5

(iii) Find the temperature of the object after 15 minutes.

(iv) Sketch the graph of T as a function of t and describe its behaviour as t continues to increase.

Section C:

Question 5: (11 marks) **START A NEW BOOKLET** Marks

(a) It is known that $\ln x + \sin x = 0$ has a root close to $x = 0 \cdot 5$. Use one application of Newton's method to obtain a better approximation (to 2 decimal places). 2

(b) The acceleration of a particle P is given by the equation $\ddot{x} = 8x(x^2 + 1)\ \text{ms}^{-2}$, where x is the displacement of P from the origin in metres after t seconds, with movement being in a straight line.

Initially the particle is projected from the origin with a velocity of 2 ms^{-1}.

(i) Show that the velocity of the particle can be expressed as $v = 2(x^2 + 1)$. 6

(ii) Hence, show that the equation describing the displacement of the particle at time t is given by $x = \tan 2t$.

(iii) Determine the velocity of the particle at time $\frac{\pi}{8}$ seconds.

(c) The arc of the curve $y = \sin^{-1} x$ between $x = 0$ and $x = 1$ is rotated about the x axis. Use Simpson's Rule with three function values to estimate the volume of the solid formed. 3

Question 6: (11 marks)

Marks

(a) The velocity v ms^{-2} of a particle moving in simple harmonic motion along the x axis is given by the expression $v^2 = 28 + 24x - 4x^2$.

(i) Between which two points is the particle oscillating?

(ii) What is the amplitude of the motion?

(iii) Find the acceleration in terms of x. 6

(iv) Find the period of the oscillation.

(v) If the particle starts from the point furthest to the right, find the displacement in terms of t.

(b) A stone is thrown from the top of a vertical cliff over the water of a lake. The height of the cliff is 8 metres above the level of the water, the initial speed of the stone is 10 ms^{-1} and the angle of projection is $\theta = \tan^{-1}\left(\frac{3}{4}\right)$ above the horizontal.

The equations of motion of the stone, with air resistance neglected, are $\ddot{x} = 0$ and $\ddot{y} = -g$.

(i) By taking the origin O as the base of the cliff, show that the horizontal and vertical components of the stone's displacement from the origin after t seconds are given by $x = 8t$ and $y = -\frac{1}{2}gt^2 + 6t + 8$. 5

(ii) Hence, or otherwise, calculate the time which elapses before the stone hits the lake and find the horizontal distance of the point of contact from the base of the cliff. (Assume g = 10 ms^{-2}.)

End of Paper

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Mathematics Extension 1

Sample Solutions

SECTION	MARKER
A	Ms Opferkuch
B	Ms Nesbitt
C	Mr Bigelow

Section A

Question 1

(a) $$\int_0^2 \frac{1}{\sqrt{16-x^2}}\,dx = \int_0^2 \frac{1}{\sqrt{4^2-x^2}}\,dx$$
$$= \left[\sin^{-1}\frac{x}{4}\right]_0^2$$
$$= \sin^{-1}\frac{1}{2}$$
$$= \frac{\pi}{6}$$

(b) (i) $$\lim_{x\to\infty}\frac{\sin 3x}{4x} = \frac{3}{4}\lim_{x\to\infty}\frac{\sin 3x}{3x}$$
$$= \frac{3}{4}\times 1$$
$$= \frac{3}{4}$$

(ii) $$\lim_{x\to\infty}\frac{\sin 3x}{\sin 7x} = \lim_{x\to\infty}\frac{\sin 3x}{3x}\times\frac{7x}{\sin 7x}$$
$$= \frac{3}{7}\lim_{x\to\infty}\frac{\sin 3x}{3x}\times\frac{7x}{\sin 7x}$$
$$= \frac{3}{7}\times 1$$
$$= \frac{3}{7}$$

(c) $$\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$$

Let $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$
$$du = \frac{1}{x}\,dx$$

$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{1}{\sqrt{1-(u)^2}}\,du$$
$$= \sin^{-1}u + C$$
$$= \sin^{-1}(\ln x) + C$$

(d) $\log_e(\sin^3 x)$

Let $u = \sin^3 x$

$$\frac{du}{dx} = 3\sin^2 x\cos x$$

Let $y = \log_e u$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$
$$= \frac{1}{u}\times 3\sin^2 x\cos x$$
$$= \frac{1}{\sin^3 x}\times 3\sin^2 x\cos x$$
$$= \frac{3\cos x}{\sin x}$$
$$\therefore \frac{dy}{dx} = 3\cot x$$

(e) $\frac{d}{dx}(\tan^{-1} x)^2$

Let $u = \tan^{-1} x$

$$\frac{du}{dx} = \frac{1}{1+x^2}$$

Let $y = u^2$

$$\frac{dy}{du} = 2u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 2u \times \frac{1}{1+x^2}$$

$$\therefore \frac{dy}{dx} = \frac{2\tan^{-1} x}{1+x^2}$$

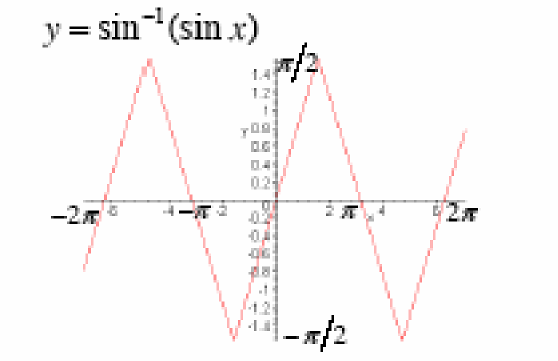
Question 2

(a) (i) $y = \sin^{-1}(\sin x)$

Domain $\{x : x \in \mathbb{R}\}$

Range $\left\{y : -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right\}$

(ii)



(b) $y = \sin^{-1}\left(\sqrt{x}\right)$

$$\sin y = \sqrt{x}$$

$$\sin^2 y = x$$

$$\therefore x = \sin^2 y$$

$$\frac{dx}{dy} = 2\sin y \cos y$$

$$= \sin 2y$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sin 2y}$$

(c) (i) $y = x\tan x - \ln(\sec x)$

Now $\frac{d}{dx} x\tan x$

Let $u = x \quad v = \tan x$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \sec^2 x$

$\therefore \frac{d}{dx}(x\tan x) = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (x)(\sec^2 x) + (\tan x)(1)$

$= x\sec^2 x + \tan x$

Now $\frac{d}{dx}\ln(\sec x)$

Let $u = \sec x \quad y = \ln u$

$= (\cos^{-1} x) \quad \frac{dy}{du} = \frac{1}{u}$

$\frac{du}{dx} = -(\cos x)^{-2}(-\sin x)$

$= \frac{\sin x}{\cos^2 x}$

$= \tan x\sec x$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{u} \times \tan x\sec x$

$= \frac{1}{\sec x} \times \tan x\sec x$

$= \tan x$

$\therefore y = x\tan x - \ln(\sec x)$

$\frac{dy}{dx} = x\sec^2 x + \tan x - \tan x$

$= x\sec^2 x$

(c)

(ii) $\int x\sec^2 x\,dx = [x\tan x - \ln(\sec x)]_0^{\frac{\pi}{4}}$

$= \left\{\frac{\pi}{4}\tan\frac{\pi}{4} - \ln(\sec\frac{\pi}{4})\right\} - \{0\tan 0 - \ln$

$= \left\{\frac{\pi}{4}(1) - \ln(\sqrt{2})\right\} - \{-\ln(1)\}$

$= \frac{\pi}{4} - \ln\sqrt{2}$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

$= \frac{\pi - 2\ln 2}{4}$

(d) $y = 10^x$

$\log_{10} y = \log_{10} 10^x$

$\log_{10} y = x\log_{10} 10$

$x = \log_{10} y$

$x = \frac{\log_e y}{\log_{e10}}$

$x = \frac{1}{\log_e 10} \times \log_e y$

$x\log_e 10 = \log_e y$

$\therefore y = e^{x\log_e 10}$

$\therefore \frac{dy}{dx} = \log_e 10 \times e^{x\log_e 10}$

when $x = 1$

$\frac{dy}{dx} = \log_e 10 \times e^{(1)\log_e 10}$

$= \log_e 10 \times 10$

$= 10\log_e 10$

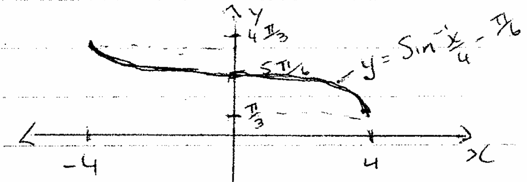
QUESTION 3

(a) Inverse $x = 4\sin(y + \frac{\pi}{6})$ $\frac{\pi}{3} \le y \le \frac{4\pi}{3}$

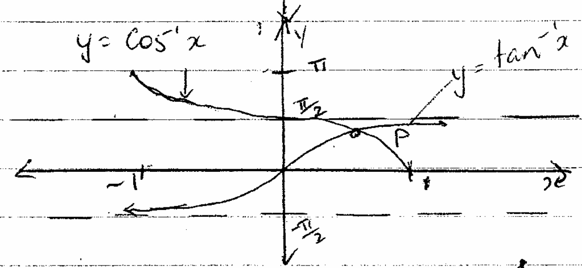
(i) $y + \frac{\pi}{6} = \sin^{-1}\frac{x}{4}$

$y = \sin^{-1}\frac{x}{4} - \frac{\pi}{6}$

Domain $-4 \le x \le 4$

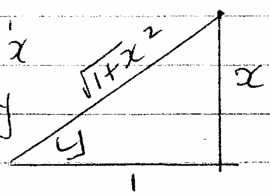


b)(i)



(ii) $\tan^{-1}x = y = \cos^{-1}x$

$\tan y = x = \cos y$



$x = \frac{1}{\sqrt{1+x^2}}$

$x^2(1+x^2) = 1$

$x^4 + x^2 - 1 = 0$

$x^2 = \frac{-1 \pm \sqrt{1+4}}{2}$

$x^2 = 0.618 \quad (x^2 > 0)$

$x = 0.79, \quad y = 0.67$ (2 d.p.)

(c) $x = \tan^{-1}(\frac{1}{4})$ $y = \tan^{-1}(\frac{3}{5})$

$\tan x = \frac{1}{4}$ $\tan y = \frac{3}{5}$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

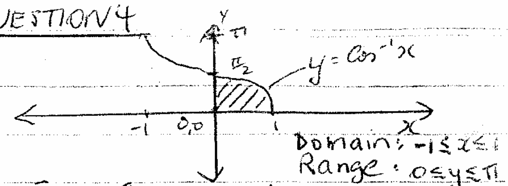
$\tan(x+y) = \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4}\cdot\frac{3}{5}} = 1$

$x + y = \frac{\pi}{4}$

$\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{3}{5}) = \frac{\pi}{4}$

QUESTION 4

a(i)



Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$

$A = \int_0^{\frac{\pi}{2}} \cos y \, dy$

$= [\sin y]_0^{\frac{\pi}{2}}$

$= 1 - 0 = 1 \, u^2$

(b) $y = \log_e 2x - 2\log_e(x-1)$

$= \frac{1}{x} - \frac{2}{x-1}$ or $-\frac{x+1}{x(x-1)}$

(c) $\frac{dT}{dt} = k(T-16)$

(i) $\frac{dt}{dT} = \frac{1}{k(T-16)}$

$t = \frac{1}{k}\log_e(T-16) + C$

$k(t-c) = \log_e(T-16)$

$T - 16 = e^{kt-kc}$

$T - 16 = Pe^{kt}$ ($P = e^{-kc}$ = constant)

$T = 16 + Pe^{kt}$ as required

(ii) $T = 16 + Pe^{kt}$

$T = 0, t = 0$ $P = -16$

$T = 16 - 16e^{kt}$

$T = 12$, $12 = 16 - 16e^{kt}$

$-4 = -16e^{10k}$

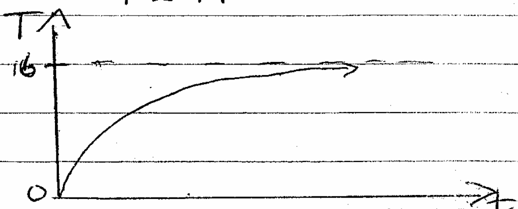
$\frac{1}{4} = e^{10k}$, $k = \frac{1}{10}\log_e\frac{1}{4}$

$k \approx -0.1386$

(iii) $t = 15$ $T = 16 - 16e^{-0.1386 \times 15}$

$T = 14°$

(iv)



as $t \to \infty$, $16e^{kt} \to 0$

$T \to 16$

QUESTION 5.

(a). Let $f(x) = \ln x + \sin x$.

$f'(x) = \frac{1}{x} + \cos x$. ($\frac{1}{2}$ for diff)

If $x_1 = 0.5$.

then $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.5 - \frac{\ln 0.5 + \sin 0.5}{\frac{1}{0.5} + \cos 0.5}$ ⇓ ① NB If calculator is in degree mode 0.73 ($\frac{1}{2}$ MARK)

$= 0.5 - -0.07427-$

$= \boxed{0.57}$ (2.D.P.). [2]

(b)(i). $\ddot{x} = 8x(x^2+1)$ and when $t=0$, $x=0$, $v=2$.

$\frac{d}{dx}(\frac{1}{2}v^2) = 8x^3+8x$

$\frac{1}{2}v^2 = 2x^4+4x^2+c$.

now $v=2$ when $x=0$

$\therefore \frac{1}{2}2^2 = 0+0+c$

$c = 2$.

$\therefore \frac{1}{2}v^2 = 2x^4+4x^2+2$

$v^2 = 4x^4+8x^2+4$

$v^2 = 4(x^4+2x^2+1)$

$= 4(x^2+1)^2$

$v = \pm 2(x^2+1)$ (now $v=2$ when $x=0$. $\therefore v \neq -2(x^2+1)$) ← must show this.

$\therefore \boxed{v = 2(x^2+1)}$ [2]

(ii) $\frac{dx}{dt} = 2(x^2+1)$

$\frac{dt}{dx} = \frac{1}{2(x^2+1)}$

$t = \frac{1}{2}\tan^{-1}x + c$

now $t=0$, when $x=0$.

$\therefore 0 = \frac{1}{2}\tan^{-1}0 + c$

$c = 0$. [2]

$\therefore t = \frac{1}{2}\tan^{-1}x$

$\therefore 2t = \tan^{-1}x \Rightarrow \boxed{x = \tan 2t}$

(iii) $\ddot{x}$ If $x = \tan 2t$

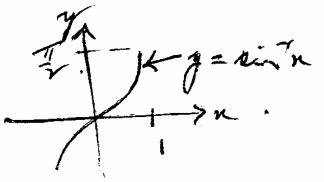
$v = 2\sec^2 2t$ ↓ [1]

At $t = \frac{\pi}{8}$

$v = 2 \times \sec^2 \frac{\pi}{4}$

$= 2 \times (\sqrt{2})^2$

$= 4\ \text{m s}^{-1}$ [2]

(c) 

$V = \pi \int_0^1 [\sin^{-1} x]^2 dx$

$\doteqdot \pi \times \frac{\frac{1}{2}}{3}\left[(\sin^{-1}0)^2 + 4[\sin^{-1}\tfrac{1}{2}]^2 + (\sin^{-1}1)^2\right]$

$= \frac{\pi}{6}\left[0^2 + 4 \times (\frac{\pi}{6})^2 + (\frac{\pi}{2})^2\right]$

$= \frac{\pi}{6}\left[0 + \frac{\pi^2}{9} + \frac{\pi^2}{4}\right]$

$= \frac{\pi}{6} \times \frac{13\pi^2}{36}$

$= \frac{13\pi^3}{216}$

$\doteqdot 1.87\ u^3$ (2 DP) [3]

QUESTION 6

(a) (i) $v^2 = 28 + 24x - 4x^2$

$= 4(7 + 6x - x^2)$

$= 4(7 - x)(1 + x)$.

Clearly $v^2 \geq 0$

$\therefore 4(7-x)(1+x) \geq 0$

$\therefore \boxed{-1 \leq x \leq 7}$ [1]

(ii) Amplitude $= \frac{7 - -1}{2} = \boxed{4}$ [1]

(iii) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}(14 + 12x - 2x^2)$

$= 12 - 4x$ [1]

$= \boxed{-4(x - 3)}$ (NB $n^2 = 4$, $n = 2$ and centre of motion is $x = 3$)

(iv) $T = \frac{2\pi}{n} = \frac{2\pi}{2}$

$= \boxed{\pi \text{ secs}}$ [1]

(v) $x = 3 + 4\cos(2t + \varepsilon)$

if $x = 7$ when $t = 0$.

$7 = 3 + 4\cos\varepsilon$.

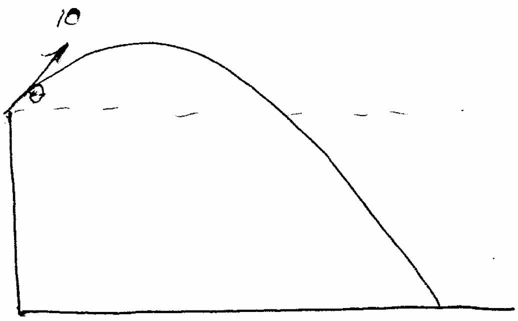
$4 = 4\cos\varepsilon$

$\cos\varepsilon = 1$

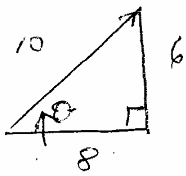
$\varepsilon = 0$ [2]

$\therefore \boxed{x = 3 + 4\cos 2t}$

(b).



$t=0, \quad x=0, \quad y=8$



NB $\theta = \tan^{-1}\left(\frac{6}{8}\right)$

(i)

$\ddot{x} = 0$

$\dot{x} = 8$

$x = 8t + c_1$

when $t=0, x=0 \therefore c_1 = 0$

$\boxed{x = 8t}$

$\ddot{y} = -g$

$\dot{y} = -gt + c_2$

Clearly $\dot{y} = 6$ when $t=0$

$\therefore c_2 = 6$

$\dot{y} = -gt + 6$

$\therefore y = -\frac{gt^2}{2} + 6t + c_3$

when $t=0, y=8 \therefore c_3 = 8$

$\therefore \boxed{y = -\frac{1}{2}gt^2 + 6t + 8}$

[3]

(ii) If $y = 0$

$-5t^2 + 6t + 8 = 0 \Rightarrow -(5t^2 - 6t - 8) = 0$

$-(5t+4)(t-2) = 0$

$t = 2, -\frac{4}{5}$

$\therefore$ 2 secs have elapsed. [1]

and $\boxed{x = 16}$ [1]