SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2005

YEAR 12

ASSESSMENT TASK #3

Mathematics Extension 1

General Instructions

- Working time – 90 minutes.
- Reading Time – 5 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work
- Hand in your answer booklets in 3 sections. Section A (Question 1), Section B (Question 2) and Section C (Question 3)

Total Marks - 80

- Attempt questions 1 – 3
- All sections are **NOT** of equal value.

Examiner: *A. Fuller*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

Total marks - 80
Attempt Questions 1 - 3
All questions are NOT of equal value

Answer each SECTION in a SEPARATE writing booklet.

Section A

Marks

Question 1 (27 marks)

(a) Evaluate $\log_2 0\cdot125$ — 1

(b) Expand $\left(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}\right)^2$ — 2

(c) Evaluate $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ — 2

(d) Sketch the graph of $y = 3\sin^{-1}\frac{x}{2}$ — 2

(e) Prove that $\log_{ab} x = \dfrac{\log_a x}{1 + \log_a b}$ — 2

(f) If $\int_0^1 \frac{1}{3 + x^2}dx = a\pi$, find the value of a — 2

(g) Differentiate $\log_e(\sin^3 x)$ writing your answer in the simplest form — 2

(h) (I) For what values of x is $\sin^{-1} x$ defined? — 1

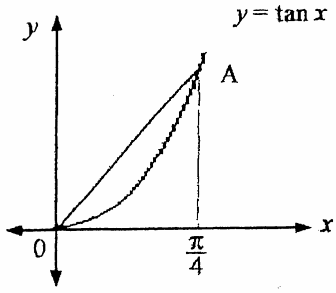
(II) Find the maximum value of $2x(1-x)$ — 1

(III) Find the range of the function given by $f(x) = \sin^{-1}[2x(1-x)]$ — 2

(i) Use the substitution $u = x^2$ to find $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}\,dx$ **3**

(j) If $y = x^n e^{ax}$, show that $\frac{dy}{dx} - ay = \frac{ny}{x}$ **3**

(k)



In the diagram, A is a point on the curve $y = \tan x$ with an x coordinate of $\frac{\pi}{4}$. The chord OA has been drawn from the origin to the point A.

Show that the area enclosed by the chord OA and the curve $y = \tan x$ between $x = 0$ and $x = \frac{\pi}{4}$ has a magnitude of $\frac{1}{8}(\pi - 4\ln 2)$ units^2 **4**

End of Section A

Section B (Use a SEPARATE writing booklet)

Marks

Question 2 (27 marks)

(a) If $y = \sec x$, prove $\dfrac{dy}{dx} = \sec x \tan x$ **2**

(b) A function is defined as $f(x) = 1 + e^{2x}$

(I) Write down the domain and the range of the function **1**

(II) Show that the inverse function can be defined as $f^{-1}(x) = \dfrac{1}{2}\ln(x-1)$ **2**

(III) On the same set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ **2**

(IV) Show that the equation of the normal to $y = f^{-1}(x)$ at the point where $f^{-1}(x) = 0$ is $2x + y - 4 = 0$ **2**

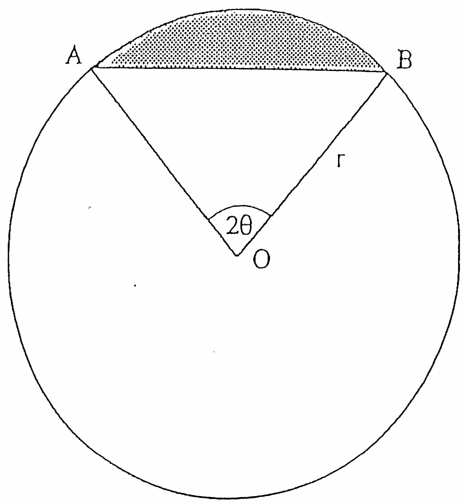
(c) If $\dfrac{dy}{dx} = 1 + y$ and when $x = 0$, $y = 2$. Show that $y = 3e^{x} - 1$ **3**

(d) Find the exact value of $\cos\left[2\sin^{-1}\dfrac{3}{4}\right]$ **3**

(e) Evaluate $\displaystyle\int \frac{3x}{\sqrt{1+x}}\,dx$ using the substitution $u = 1 + x$ **3**

(f) If $f(x) = a\cos^{-1}(bx)$, evaluate a and b if $f(0) = 2$ and $f'(0) = 2$. **3**

(g)



The diagram above shows a shaded segment which subtends an angle of 2θ radians at the centre O of a circle with radius r. Given that the perimeter of the shaded segment equals twice the diameter of the circle

(I) Show that $\sin\theta = 2 - \theta$ **2**

(II) Show that the equation $\sin\theta + \theta - 2 = 0$ has a root that lies between $\theta = 1$ and $\theta = 1\cdot 5$ **1**

(III) Use one application of Newton's method with an initial approximation of $\theta = 1\cdot 25$ to obtain a better approximation of the root of the equation $\sin\theta + \theta - 2 = 0$ **2**

(IV) Using the result found in (III) find to the nearest degree the size of $\angle AOB$ **1**

End of Section B

Section C (Use a SEPARATE writing booklet)

Marks

Question 3 (26 marks)

(a) The velocity v of a particle moving along the x axis starting from $x = 1 \cdot 8$ is given by $v = e^{-2x}\sqrt{2x^2 - 6}$, $x \geq 1 \cdot 8$ where x is the displacement of the particle from the origin.

(I) Show that the acceleration a of the particle in terms of its displacement can be expressed by $a = -2e^{-4x}\left(2x^2 - x - 6\right)$ **2**

(II) Hence, find the displacement of the particle at which the maximum speed occurs. **1**

(III) Show that the time T in seconds taken by the particle to move from $x = 2$ to $x = 3$ can be expressed as **2**

$$T = \int_2^3 \frac{e^{2x}}{\sqrt{2x^2 - 6}}\,dx$$

(IV) Use Simpson's rule with three function values to obtain an approximate value for T. **2**

(b) A water tank is generated by rotating the curve $y = \frac{x^4}{16}$ around the y-axis.

(I) Show that the volume of water, V as a function of its depth h, is given by $V = \frac{8}{3}\pi h^{\frac{3}{2}}$ **2**

(II) Water drains from the tank through a small hole at the bottom. The rate of change of the volume of water in the tank is proportional to the square root of the water's depth. **3**

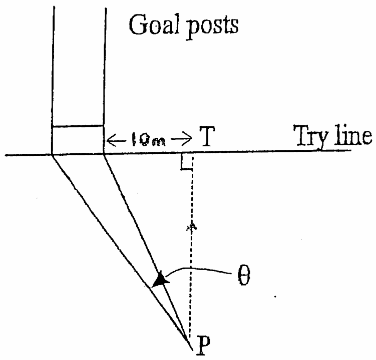
Use this fact to show that the water level in the tank falls at a constant rate.

(c) Given $y = \cos^{-1}(\sin x)$

(I) Show that $\frac{dy}{dx}$ has two values **3**

(II) Hence, or otherwise sketch the graph of $y = \cos^{-1}(\sin x)$ for $-\pi \le x \le \pi$ **2**

(e) In rugby league, teams score points by placing the ball on (or over) the try line at the end of the field. A kicker may then convert the try by taking the ball back at right angles from the point T on the try line where the try was scored and attempt to kick the ball between the goal posts which are 6 metres apart.



In the diagram above, a try has been scored 10 metres to the right of the goal posts. The kicker has brought the ball back x metres to a point P to attempt the conversion. The kicker wants to maximise θ, the angle of his view of the goal posts.

(I) Show that $\tan\theta = \dfrac{6x}{160 + x^2}$. **3**

(II) Letting $E = \tan\theta$, find the value of x for which E is a maximum. **2**

(III) Hence show that the maximum angle, θ, is given by $\theta = \tan^{-1}\left(\dfrac{3}{\sqrt{160}}\right)$ **2**

Question (e) continued over the page

(IV) Find the maximum value of θ (to the nearest minute) and the corresponding value of x (to the nearest centimetre). **2**

End of paper

SYDNEY BOYS HIGH SCHOOL
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2005
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #3

Mathematics Extension 1

Sample Solutions

Section	Marker
A	AMG
B	FN
C	EC

Question 1

(a) $\log_2 0{\cdot}125 = \log_2\left(\frac{1}{8}\right)$

$= \log_2 (2^{-3})$

$= -3$

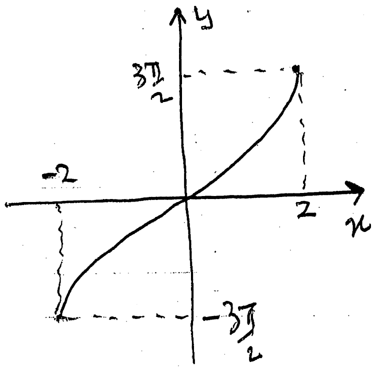
(b) $\left(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}\right)^2 = e^{x} + e^{-x} - 2$

(c) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$= \frac{\pi}{4} + \left(-\frac{\pi}{6}\right)$

$= \frac{\pi}{12}$

(d)



(e) $\log_{ab} x = \frac{\log_a x}{\log_a ab}$

$= \frac{\log_a x}{\log_a a + \log_a b}$

$= \frac{\log_a x}{1 + \log_a b}$ QED

(f) $\int_0^1 \frac{1}{3+x^2}\,dx = \left[\frac{1}{\sqrt{3}} \tan^{-1}\frac{x}{\sqrt{3}}\right]_0^1$

$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{6} - 0\right)$

$= \frac{\pi}{6\sqrt{3}}$

$\therefore a = \frac{1}{6\sqrt{3}} = \frac{\sqrt{3}}{18}$

(g) $\frac{d}{dx} \ln(\sin^3 x)$

$= \frac{1}{\sin^3 x} \times 3\sin^2 x \times \cos x$

$= 3\cot x$

(h)(i) $-1 \le x \le 1$

(ii) $2x(1-x) = 2x - 2x^2$

$= -2(x^2 - x)$

$= -2(x^2 - x + \frac{1}{4} - \frac{1}{4})$

$= -2(x - \frac{1}{2})^2 + \frac{1}{2}$

$\therefore$ Max Value $= \frac{1}{2}$

(iii) Range:

$2x - 2x^2 = -1$ for $x = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$

Range: $-\frac{1}{2} \le y \le \frac{\pi}{6}$

Q1 (cont'd)

(i) $I = \int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^4}}\,dx$

$u = x^2$ When $x = 0, u = 0$

$du = 2x\,dx$ $\quad x = \frac{1}{\sqrt{2}}, u = \frac{1}{2}$

$I = \frac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}}$

$= \frac{1}{2}\left[\sin^{-1} u\right]_0^{1/2}$

$= \frac{1}{2}\left[\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right]$

$= \frac{1}{2}\left[\frac{\pi}{6} - 0\right]$

$= \frac{\pi}{12}$

(j) $y = x^n e^{ax}$

$\frac{dy}{dx} = x^n . a e^{ax} + e^{ax} n x^{n-1}$

$= e^{ax}(ax^n + nx^{n-1})$

LHS $= \frac{dy}{dx} - ay = nx^{n-1}e^{ax}$

RHS $= \frac{ny}{x}$

$= \frac{nx^{n}e^{ax}}{x}$

$= nx^{n-1}e^{ax}$

$=$ LHS

QED

(k) Point A is $\left(\frac{\pi}{4}, 1\right)$

Line OA: $m = \frac{1}{\pi/4}$

$= \frac{4}{\pi}$

$\therefore y = \frac{4}{\pi}x$

Area between chord & line:

$A = \int_0^{\pi/4} \left(\frac{4}{\pi}x - \tan x\right) dx$

$= \int_0^{\pi/4} \left(\frac{4}{\pi}x - \frac{\sin x}{\cos x}\right) dx$

$= \left[\frac{2}{\pi}x^2 + \ln(\cos x)\right]_0^{\pi/4}$

$= \left(\frac{\pi}{8} + \ln\frac{1}{\sqrt{2}}\right) - (0 + \ln 1)$

$= \frac{\pi}{8} - \frac{1}{2}\ln 2$

$= \frac{1}{8}(\pi - 4\ln 2)\ u^2$

SECTION B

a) $y = \dfrac{1}{\cos x}$

$= \dfrac{\cos x \times 0 - 1 \times -\sin x}{\cos^2 x}$ *(quot. rule)*

$= \dfrac{\sin x}{\cos^2 x}$

$= \sec x \tan x$

b)(i) $f(x) = 1 + e^{2x}$

domain: all real x

range : $f(x) > 1$

(ii) let $y = f(x) = 1 + e^{2x}$

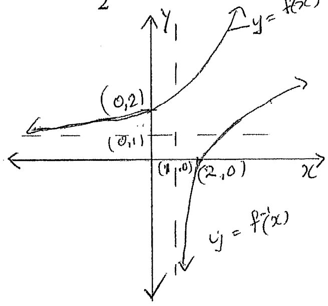
$y - 1 = e^{2x}$

$\log(y-1) = 2x$

$x = \frac{1}{2}\log(y-1)$

$f^{-1}(x) = \frac{1}{2}\log(x-1)$

(iii)



(iv) $\dfrac{d}{dx}\left[\frac{1}{2}\log(x-1)\right] = \frac{1}{2}\left(\frac{1}{x-1}\right)$

slope of tangent = $\dfrac{1}{2x-2}$

slope of normal = $-(2x-2)$

= -2 when $x = 2$

eqn. of normal is $y - 0 = -2(x-2)$

as $f^{-1}(x) = 0$ *when* $x = 2$

eqn. of normal is $2x + y - 4 = 0$

c) $\dfrac{dy}{dx} = 1 + y$, $\dfrac{dx}{dy} = \dfrac{1}{1+y}$

integrating both sides with respect to y

$x = \ln(1+y) + c$

$0 = \ln 3 + c$ *as* $x = 0$ *when* $y = 2$

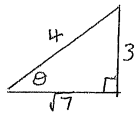
$x = \ln(1+y) - \ln 3$

$x = \ln\left(\dfrac{1+y}{3}\right)$

$e^x = \left(\dfrac{1+y}{3}\right)$

$y = 3e^x - 1$

(d) $\cos\left[2\sin^{-1}\left(\frac{3}{4}\right)\right]$ let $\vartheta = \sin^{-1}\left(\frac{3}{4}\right)$



$\cos 2\vartheta = 2\cos^2\vartheta - 1$

$= 2 \times \frac{7}{16} - 1$ (diagram)

$\cos\left[2\sin^{-1}\left(\frac{3}{4}\right)\right] = -\frac{2}{16} = -0 \cdot 125$

(e)

$\displaystyle\int \frac{3x}{\sqrt{1+x}}dx$ $\qquad u = 1 + x,\ x = u - 1$

$\dfrac{du}{dx} = 1, \quad dx = du$

$= \displaystyle\int \frac{3u-3}{\sqrt{u}}du$

$= \displaystyle\int (3u^{\frac{1}{2}} - 3u^{-\frac{1}{2}})du$

$= 2u^{\frac{3}{2}} - 6u^{\frac{1}{2}} + c$

$= 2\sqrt{(1+x)^3} - 6\sqrt{1+x} + c$

(f) $f(x) = a\cos^{-1} bx, f(0) = 2, f'(0) = 2$

$a\cos^{-1}(0) = 2,\ \cos^{-1}(0) = \frac{2}{a}$

$\frac{\pi}{2} = \frac{2}{a},\quad a = \frac{4}{\pi}$

$f'(x) = -\dfrac{ab}{\sqrt{1-b^2x^2}}$

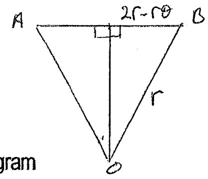
$f'(0) = -ab = 2$

$-\frac{4}{\pi}b = 2,\ b = -\frac{\pi}{2},\ a = \frac{4}{\pi}$

(g)(i) perimeter = AB + arcAB

$4r = AB + r \times 2\vartheta$

$AB = 4r - 2r\vartheta$



$\sin\vartheta = \dfrac{2r - r\vartheta}{r}$ from diagram

$\sin\vartheta = 2 - \vartheta$

(ii)

$\sin(1) + 1 - 2 = -0 \cdot 1585, \ < 0$

$\sin(1 \cdot 5) + 1 \cdot 5 - 2 = 0 \cdot 497, \ > 0$

as f(x) is contiuous, the sign change means there must be a root between $\vartheta = 1$ *and* $\vartheta = 1 \cdot 5$.

(iii) $f'(\vartheta) = \cos\vartheta + 1$

$a_1 = 1 \cdot 25 - \dfrac{\sin(1 \cdot 25) + 1 \cdot 25 - 2}{\cos(1 \cdot 25) + 1}$

approx. of root = $1 \cdot 0987$

(iv) $\vartheta \approx 1 \cdot 0987$

$2\vartheta \approx 2 \cdot 1974^c$

$2\vartheta \approx 125°54'$

$\angle AOB = 126°$ (*nearest* degree)

[24 marks]

Question (3).

(i) $V = e^{-2x}(2x^2-6)^{\frac{1}{2}}$

$\frac{V^2}{2} = \frac{e^{-4x}}{2}(2x^2-6) = e^{-4x}(x^2-3)$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}V^2\right)$

$a = (2x^2-6)(-2e^{-4x}) + \frac{e^{-4x}}{2}(4x)$.

$a = 2xe^{-4x} - 4e^{-4x}(x^2-3)$

$a = -2e^{-4x}(2x^2-x-6)$

[2]

(ii) max speed when $\ddot{x} = 0$.

Now, $-2e^{-4x} < 0, \forall x$

∴ $2x^2 - x - 6 = 0$

$(2x+3)(x-2) = 0$

∴ $x = -\frac{3}{2}, 2$. [1]

∵ $x \geq 1.8$ ∴ $x = 2$.

(iii) $V = \frac{dx}{dt} = e^{-2x}\sqrt{2x^2-6}$

∴ $\frac{dt}{dx} = \frac{e^{2x}}{\sqrt{2x^2-6}}$.

$T = \int_2^3 \frac{e^{2x}}{\sqrt{2x^2-6}}\,dx$. [2]

(iv) $\int_2^3 \frac{e^{2x}}{\sqrt{2x^2-6}}\,dx$ (1)

$\doteq \frac{h}{3}\left[(y_1+y_3)+4y_2\right]$

$h = \frac{1}{2}$ (1)

$= \frac{1}{6}\left[(y_1+y_3)+4y_2\right]$.

$= \frac{1}{6}\left[\frac{e^4}{\sqrt{2}} + \frac{e^6}{2\sqrt{3}} + 4\left(\frac{e^5}{\sqrt{6.5}}\right)\right]$

$= \frac{1}{6}\left[38.61 + 116.46 + 232.85\right]$

$= 64.65$ [2]

(b)

(i) $V = \pi\int_0^h x^2\,dy$

$x^4 = 16y$, $x^2 = 4y^{\frac{1}{2}}$ (1)

∴ $V = 4\pi\int_0^h y^{\frac{1}{2}}\,dy$

$= 4\pi\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^h$

$= \frac{8\pi}{3}h^{\frac{3}{2}}$ [2]

(ii) $\frac{dV}{dt} \propto \sqrt{h}$ (1)

∴ $\frac{dV}{dt} = -k\sqrt{h}$

Where k is a constant

Now, $\frac{dh}{dt} = \frac{1}{\left(\frac{dV}{dh}\right)} \times \frac{dV}{dt}$ (1)

$= \frac{1}{4\pi\sqrt{h}} \times -k\sqrt{h}$

$= -\frac{k}{4\pi}$ (a constant) [3]

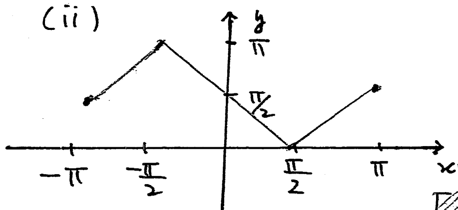
(c) (i) $y = \cos^{-1}(\sin x)$

$\frac{dy}{dx} = \frac{-\cos x}{\sqrt{1-\sin^2 x}}$

$= \frac{-\cos x}{|\cos x|}$ [3]

$= \pm 1$

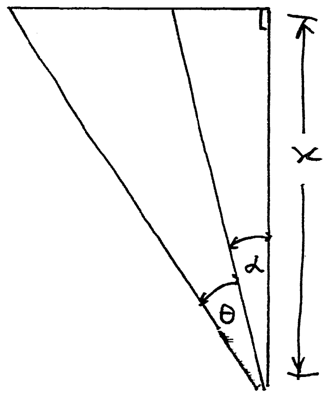
(ii)



[2]

(d)

(i)



$$\frac{16}{x} = \tan(\alpha + \theta)$$

$$\frac{10}{x} = \tan\alpha$$

$$\tan(\alpha+\theta) = \frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta}$$ (1)

$$\therefore \quad \frac{16}{x} = \frac{\frac{10}{x} + \tan\theta}{1 - \frac{10}{x}\tan\theta} = \frac{\left(\frac{10 + x\tan\theta}{x}\right)}{\left(\frac{x - 10\tan\theta}{x}\right)}$$ (1)

$$10x + x^2\tan\theta = 16x - 160\tan\theta$$

$$(160 + x^2)\tan\theta = 6x$$

$$\therefore \quad \tan\theta = \frac{6x}{x^2+160}.$$

[3]

(ii) Let $E = \tan\theta$.

(ii) $$\frac{dE}{dx} = \frac{6(160+x^2) - 6x(2x)}{(160+x^2)^2}$$ (1)

$$= \frac{6(160 - x^2)}{(160+x^2)^2}$$

[2]

$$\frac{dE}{dx} = 0, \implies x^2 = 160, \quad x = \sqrt{160}$$

x	12	$4\sqrt{10}$	$4\sqrt{11}$
$\frac{dE}{dx}$	+	0	−

$+ \; 0 \; -$

$\therefore \; x = \sqrt{160}$

(1) for maximum E

(iii)

$$\tan\theta = \frac{6(4\sqrt{10})}{160+160}$$ (1)

$$= \frac{24\sqrt{10}}{320}$$

$$= \frac{3\sqrt{10}}{40}$$

[2]

$$= \frac{3\sqrt{10}}{\sqrt{1600}} = \frac{3}{\sqrt{160}}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{3}{\sqrt{160}}\right).$$