SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2006
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #3

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each **section** is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 72

- Attempt questions 1 – 6
- All questions are of equal value.

Examiner: *A.M.Gainford*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\, dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \ln\left(x + \sqrt{x^2 - a^2}\right)\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x,\ x > 0$

Section A

(Start a new booklet.)

Question 1. (12 marks)

Marks

(a) In how many ways can a committee of five be chosen from four women and six men, given that at least two must be women. **2**

(b) AB and AC are two chords of a circle on opposite sides of the centre, O. P and Q are the midpoints of AB and AC respectively. Prove that A, P, O and Q are concyclic. (Give reasons for each step.)

(c) Use the substitution $u = 1 - x$ to find $\int_0^1 x\sqrt{1-x}\,dx$. **4**

(d) Write down the inverse function of $y = \sqrt{x-1}$ as a function of x and state its domain. **2**

(e) State the exact value of: **2**

(i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(ii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Section continued overleaf.

Question 2 (12 marks)

Marks

(a) Differentiate (i) $\tan^{-1} 2x$. **6**

(ii) $\sin^{-1}\left(\frac{x}{3}\right) + \cos^{-1}\left(\frac{x}{3}\right)$

(b) Find a primitive of: **6**

(i) $\frac{1}{\sqrt{4-x^2}}$

(ii) $\frac{1}{1+4x^2}$

(iii) $\frac{1}{2x+1}$

End of Section A

Section B
(Start a new booklet.)

Question 3. (12 marks)

Marks

(a) (i) State the domain and range of the function $f(x) = \sin^{-1}(-x)$. **6**

(ii) Sketch the graph of $y = f(x)$.

(iii) Find the gradient of the tangent to the curve at the point where it crosses the y-axis.

(b) Use the substitution $u = \log_e x$ to evaluate $\int_e^{e^2} \frac{1}{x\ln x}\,dx$. **3**

(c) A particle starts from O with acceleration $a = \cos t$ and velocity $0\cdot5$ m/sec. **3**

(i) Find its position when $t = 3\pi$ sec.

(ii) Find the total distance traveled in the first 3π seconds.

(iii) Sketch the graph of position against time for $0 \le t \le 3\pi$.

Question 4 (12 marks)

Marks

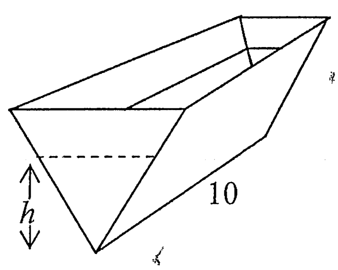
(a) (i) Differentiate e^{-x^2}. **6**

(ii) Hence show that $\int_0^1 xe^{-x^2}\,dx = \frac{1}{2}\left(1 - \frac{1}{e}\right)$.

(b) A large irrigation trough in the shape of a triangular prism with base angle $60°$ is being filled with water at a constant rate of $2\ m^3/\text{min}$. **6**

(i) Write an equation for the volume of water in the trough in terms of h, the depth of the water in metres.

(ii) Find the rate of change of depth of the water when the depth is 2 m.



End of Section B

Section C

(Start a new booklet.)

Question 5: (12 marks)

Marks

(a) The area bounded by the curve $y = \frac{b}{a}\sqrt{a^2 - x^2}$ (where a and b are constants), and the x-axis, is rotated about the x-axis. **4**

Find the volume of the solid of revolution so formed.

(b) Find the exact value of $\int_0^{\frac{\pi}{3}} \sin^2 x \, dx$. **4**

(c) (i) Find the derivative of $x \ln x - x$. **4**

(ii) Hence evaluate $\int_1^2 \ln x^2 dx$.

Question 6: (12 marks)

Marks

(a) A rectangular table has eight seats, one on each end, and three on each side. Calculate the number of distinct ways that eight people may be seated around the table, if: **3**

(i) there are no restrictions,

(ii) two particular people must not occupy end seats.

(b) Consider the curves $y = x^3$ and $y = \sqrt[3]{x}$. **3**

(i) Sketch the graphs of the two curves on the same axes, in the domain $0 \le x \le 1 \cdot 5$.

(ii) Determine the area of the region bounded by the curves in the stated domain.

Question continued overleaf.

Marks

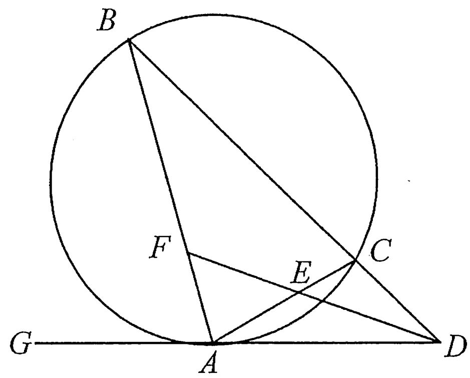
(c) (i) Show that $x = 2 \cdot 1$ is an approximate solution of the equation $x + \ln x = 3$. **3**

(ii) Use one application of Newton's Method to find a better solution.

(d) In the diagram at right AD is a tangent to the circle, and DF bisects $\angle ADB$. **3**

(i) Copy the diagram to your answer sheet.

(ii) Prove that $AF = AE$.



End of Section C

This is the end of the paper.

Solution : Section A

Q (1) /12

4 W	6 M
2 W	3 M
3 W	2 M
4 W	1 M

[2]

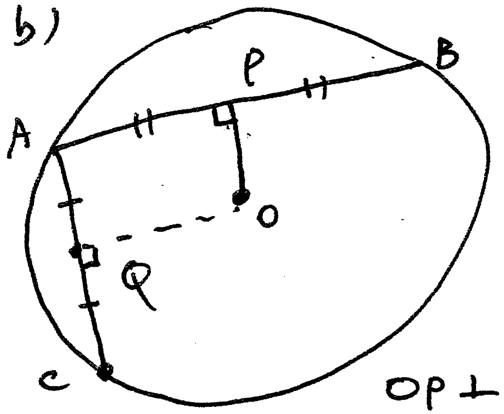
- At least 2 W
- Choose 5 people.

∴ No. of ways

$= \binom{4}{2}\binom{6}{3} + \binom{4}{3}\binom{6}{2} + \binom{4}{4}\binom{6}{1}$

$= 120 + 60 + 6 = \boxed{186}.$

(b)



OP ⊥ AB [2]

- Line from centre of the circle to the mid pt of a chord is perp. to the chord.

Similarly, OQ ⊥ AC

∴ APOQ is a cyclic quad (opp angles are supplementary)

i.e A, P, O, Q are concyclic.

(c) Let $u = 1 - x$, $du = -dx$

When $x = 0$, $u = 1$, $x = 1$, $u = 0$

∴ $\int_0^1 x\sqrt{1-x}\,dx$

$= \int_1^0 (1-u)\sqrt{u}\,(-du)$

$= \int_0^1 (u^{\frac{1}{2}} - u^{3/2})\,du$

$= \left[\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^1$

$= 4/15$ [4]

(d) $f(x) = \sqrt{x-1}$ [2]

$d_f : x \geq 1,\ y \geq 0$

$x = \sqrt{y-1}$ ∴ $x^2 = y - 1$

$\Longrightarrow \boxed{f^{-1}(x) = x^2 + 1}$, $x \geq 0$, $y \geq 1$

(e) (i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. (1) [2]

(ii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\tan^{-1}\frac{1}{\sqrt{3}}$

$= -\frac{\pi}{6}.$ (1)

Q (2) /12

(a) (i) $\frac{d}{dx}\tan^{-1}(2x)$

$= \frac{2}{1+4x^2}$ $\quad \frac{1}{2(\frac{1}{4}+x^2)}$ [2]

(ii) $\frac{d}{dx}\left[\left(\sin^{-1}\frac{x}{3}\right) + \left(\cos^{-1}\frac{x}{3}\right)\right]$

$= \frac{1}{\sqrt{9-x^2}} - \frac{1}{\sqrt{9-x^2}}$

$= 0$ [4]

(b) (i) $\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2} + c$ [2]

(ii) $\int \frac{dx}{1+4x^2} = \frac{1}{4}\int \frac{dx}{x^2+\frac{1}{4}}$

$= \frac{1}{2}\tan^{-1}(2x) + c$ [2]

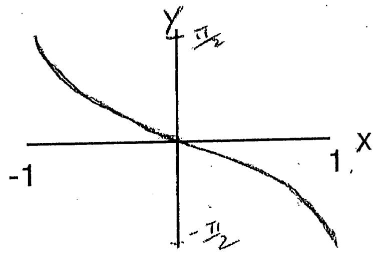
(iii) $\int \frac{dx}{2x+1} = \frac{1}{2}\int \frac{2\,dx}{2x+1}$

$= \frac{1}{2}\ln|2x+1| + c.$ [2]

QUESTION 3

(a) (i) D: $-1 \le x \le 1$, R: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(ii)



(iii) $y = \sin^{-1}(-x)$

$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$x = 0$, grad. of tangent $= -1$

(b) $u = \log x$, $\frac{du}{dx} = \frac{1}{x}$, $dx = x.du$

$when\ x = e^2, u = 2,\ \ when\ x = e, u = 1$

$\int_1^2 \frac{1}{x.u} x.du = \int_1^2 \frac{1}{u} du = [\ln u]_1^2$

$= \ln 2 - \ln 1 = \ln 2$

(c) $a = \cos t = \frac{dv}{dt}$

$v = \sin t + c$

$t = 0,\ v = 0.5,\ c = 0.5$

$v = \sin t + 0.5 = \frac{dx}{dt}$

$x = -\cos t + 0.5t + k$

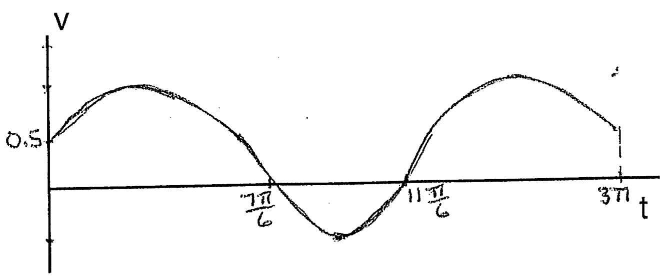
when $t = 0$ $x = 0$, $k = \cos 0 = 1$

$x = -\cos t + 0.5t + 1$

$t = 3\pi,\ x = \left(2 + 3\frac{\pi}{2}\right)$m

(d) (ii) distance travelled $= \int_0^{3\pi} (\sin t + 0.5) dt$

v=sint+0.5



$= 2\left[0 \cdot 5t - \cos t\right]_0^{\frac{7\pi}{6}} + \left|(0 \cdot 5t - \cos t)\right|_{\frac{7\pi}{6}}^{3\pi}$

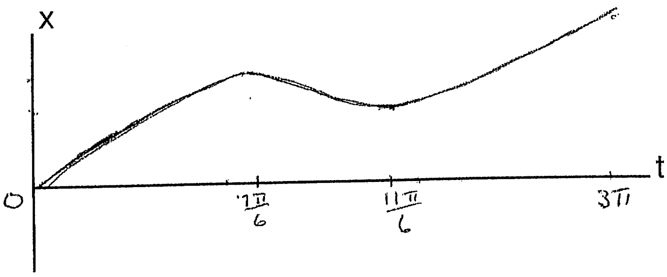
$= 2\left[\frac{7\pi}{12} + \frac{\sqrt{3}}{2} - (0-1)\right] + \left|\frac{11\pi}{12} - \frac{\sqrt{3}}{2} - \left(\frac{7\pi}{12} + \frac{\sqrt{3}}{2}\right)\right|$

$= \left(\frac{5\pi}{6} + 2\sqrt{3} + 2\right) m$

(iii) $\frac{dx}{dt} = \sin t + 0 \cdot 5$

turning points $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

x=-cost+0.5t+1



QUESTION 4

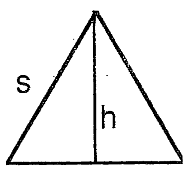
(a) (i) $\frac{d\left(e^{-x^2}\right)}{dx} = -2xe^{-x^2}$

(ii) $\int_0^1 -2xe^{-x^2} dx = \left[e^{-x^2}\right]_0^1$

$\int_0^1 xe^{-x^2} dx = -\frac{1}{2}\left[e^{-x^2}\right]_0^1$

$= \frac{1}{2}\left(1 - \frac{1}{e}\right)$

(b) (i)



$s = \frac{h}{\sin 60^\circ} = \frac{2h}{\sqrt{3}}$

v=area of triangle × 10

$= \frac{1}{2} sh \times 10 = \frac{1}{2} \times \frac{2h}{\sqrt{3}} \times h \times 10$

$v = \frac{10h^2}{\sqrt{3}}$

(ii) $\frac{dv}{dh} = \frac{20h}{\sqrt{3}}$, $\frac{dh}{dv} = \frac{\sqrt{3}}{20h}$

$\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$

$\frac{dv}{dt} = 2$

$\frac{dh}{dt} = \frac{\sqrt{3}}{20h} \times 2$

When h=2, the rate of change of depth is

$\frac{\sqrt{3}}{20}$ m/min

SECTION C

5) a) $y = \frac{b}{a}\sqrt{a^2 - x^2}$

Find x-intercepts (let y=0)

$$\frac{b}{a}\sqrt{a^2 - x^2} = 0$$

$$\sqrt{a^2 - x^2} = 0$$

$$a^2 - x^2 = 0$$

$$x^2 = a^2$$

$$x = \pm a$$

Let $f(x) = \frac{b}{a}\sqrt{a^2 - x^2}$

$$f(-x) = \frac{b}{a}\sqrt{a^2 - (-x)^2}$$

$$= \frac{b}{a}\sqrt{a^2 - x^2}$$

$f(x) = f(-x)$ ∴ Even. (symmetrical about y-axis)

$$V = \pi \int_{-a}^{a} y^2 \, dx$$

$$V = \pi \int_{-a}^{a} \left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 dx$$

$$V = 2\pi \frac{b^2}{a^2} \int_0^a (a^2 - x^2)\, dx$$

$$V = \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$V = \frac{2\pi b^2}{a^2}\left[a^3 - \frac{a^3}{3} - 0\right]$$

$$V = \frac{2\pi b^2}{a^2}\left[\frac{2a^3}{3}\right]$$

$$V = \frac{4\pi b^2 a}{3} \text{ cubic units}$$

b) $\int_0^{\frac{\pi}{3}} \sin^2 x \, dx$

$= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x \right) dx$

$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x \right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{4}\sin\frac{2\pi}{3} - (0)$

$= \frac{\pi}{6} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$

c) i) $y = x\ln x - x$

$\frac{dy}{dx} = x \cdot \frac{1}{x} + 1 \cdot \ln x - 1$

$\frac{dy}{dx} = 1 + \ln x - 1$

$\frac{dy}{dx} = \ln x$

ii) $\int_1^2 \ln x^2 \, dx$

$= \int_1^2 2\ln x \, dx$

$= 2\int_1^2 \ln x \, dx$

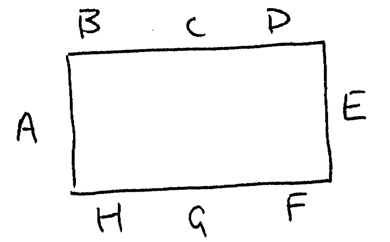
$= 2\left[x\ln x - x \right]_1^2$

$= 2\left(2\ln 2 - 2 - (1 \cdot \ln 1 - 1)\right)$

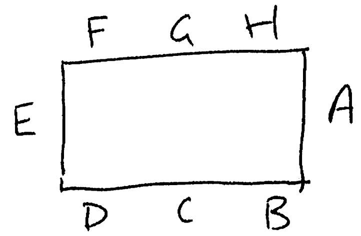
$= 2(2\ln 2 - 1)$

$= 4\ln 2 - 2$

6) a) i)



would be the same as



so we must divide by 2

ie number of distinct ways is $\frac{8!}{2} = \frac{40320}{2}$

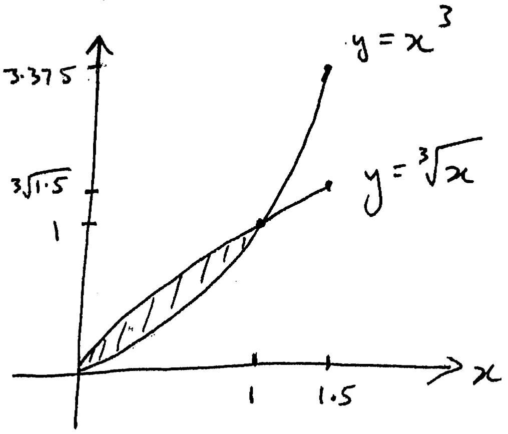
$= 20160$

ii) if two must not occupy the end seat we are choosing 2 from 6 (not interested in order because of symmetry)

then choosing 6 from 6 (order matters)

$^6C_2 \times {}^6P_6 = 10800$

b) i)



ii) Area bounded in the domain (shaded)

$A = \int_0^1 (x^{\frac{1}{3}} - x^3)\, dx$

$= \left[\frac{3}{4} x^{\frac{4}{3}} - \frac{x^4}{4} \right]_0^1$

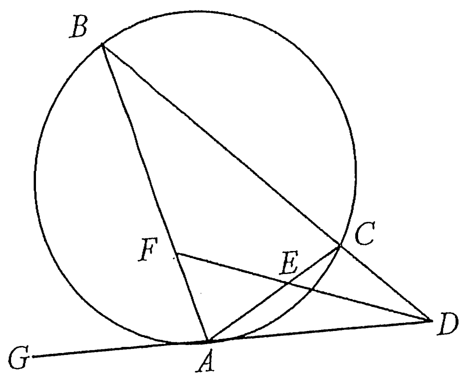
$= \left[\frac{3}{4}(1)^{\frac{4}{3}} - \frac{(1)^4}{4} \right] - [0]$

$= \frac{3}{4} - \frac{1}{4}$

$= \frac{1}{2}$ square units

c) (omitted)

d) i)



ii) let $\angle FDB = x$

$\angle FDA = x$ (DF bisects $\angle ADB$)

let $\angle CAD = y$

$\angle ABD = y$ (angle in the alternate segment equals the angle between the chord and tangent)

$\angle AFD = x+y$ (the exterior angle of $\triangle BFD$ is equal to the sum of the two opposite interior angles)

$\angle FEA = x+y$ (the exterior angle of $\triangle EAD$ is equal to the sum of the two opposite interior angles)

$\therefore$ $\triangle AFE$ is isosceles ($\angle AFD = \angle FEA$)

$\therefore$ $AF = AE$ (sides opposite equal angles in isosceles triangle).