SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2010

YEAR 12

ASSESSMENT TASK #3

Mathematics Extension 1

General Instructions

- Working time – 90 Minutes
- Reading Time – 5 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each **Section** is to be returned in a separate booklet.
- All necessary working should be shown in every question if full marks are to be awarded.
- Full marks may not be awarded for untidy or badly arranged work.

Total Marks – 72

- Attempt questions 1 – 6
- All questions are NOT of equal value.
- Unless otherwise directed give your answers in simplest exact form.

Examiner: *A.M.Gainford*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right)\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Section A
(Start a new booklet.)

Question 1. (12 marks)

Marks

(a) Prove the following identity: **2**

$$\frac{\sin A}{\cos A+\sin A}+\frac{\sin A}{\cos A-\sin A}=\tan 2A$$

(b) Find: $\lim\limits_{x\to 0}\frac{\sin 3x}{x}$ **1**

(c) Use the substitution $u = 2x - 1$ to find **4**

$$\int_{\frac{1}{2}}^{1} 4x(2x-1)^3\,dx$$

(d) The sector AOB of a circle centre O and radius r has an area of $\frac{3\pi}{4}$ cm^2. If the arc AB subtends an angle of $\frac{\pi}{6}$ at O, find the length of the arc AB. **2**

(e) Give the solutions of the equation $\sin\left(\theta-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$ in $0 \le \theta \le 2\pi$. **3**

Question 2. (12 Marks)

(a) Differentiate the following with respect to x: **6**

(i) $x^2\tan 2x$

(ii) $\cos^{-1}3x$

(iii) $\log_e(\cos x)$

(b) Find an indefinite integral of each of the following (with respect to x): **6**

(i) $\frac{1}{\sqrt{4-x^2}}$

(ii) $\sin(1-3x)$

(iii) $\sin x\cos^3 x$

Section B
(Start a new booklet.)

Question 3. (12 marks)

(a) (i) State the domain and range of the function $f(x) = 3\sin^{-1}\left(\frac{x}{2} - 1\right)$. **5**

(ii) Sketch the graph of $y = f(x)$.

(iii) Evaluate $f(1)$.

(b) Use the substitution $u = \cot x$ to evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2 x \operatorname{cosec}^2 x \, dx$. **3**

(c) The equation $x^3 - 103x + 102 \cdot 5 = 0$ has a root near $x = 1$. **2**
Take $x = 1$ as a first approximation and use Newton's method once to obtain a closer approximation to this root.

(d) A particle moves along a straight line so that its distance from a fixed point O at a time t is given by $x = 2\cos\left(\frac{t}{4}\right)$. **2**

What is the exact value of the acceleration of the particle when $t = \frac{4\pi}{3}$?

Section continues overleaf

Question 4. (12 marks)

Marks

(a) A particle P moves along the x-axis so that at time t seconds it is x cm from the origin O and its velocity is v cm/s. Initially the particle is at rest at the origin. **6**

(i) If the acceleration of P is given by $\ddot{x} = 4(40 - x)$ cm/s^2, use $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to show $v^2 = 4(80x - x^2)$.

(ii) Prove that P moves in the interval $0 \le x \le 80$.

(iii) Find the maximum velocity of the particle, and where the maximum occurs.

(b) The inside of a vessel used for water has the shape of a solid of revolution obtained by the rotation of the parabola $9y = 8x^2$ about the y-axis. **6**
The depth of the vessel is 8 cm.

(i) Prove that the volume of water, when the depth is h cm, is $\frac{9}{16}\pi h^2$ cm^3.

(ii) If water is poured into the vessel at a rate of 20 cm^3/s, find the rate at which the water is rising when the depth is 4 cm.

Section C
(Start a new booklet.)

Question 5. (12 marks)

Marks

(a) The digits 1, 2, 3, 4, 5 are arranged in a row. **4**

(i) In how many ways can this be done without restriction?

(ii) In how many of these arrangements will there be two or more odd numbers together?

(b) Find the exact value of $\int_0^{\frac{\pi}{8}} \sin 2x \cos^2 2x dx$. **2**

(c) Find $\cos\theta$ if $\theta = \cos^{-1}\frac{24}{25} - \sin^{-1}\frac{15}{17}$. **2**

(d) A cylindrical hole of radius c is bored symmetrically through a sphere of radius a. Show that the remaining volume is $\frac{4}{3}\pi(a^2 - c^2)^{\frac{3}{2}}$. **4**

Section continues overleaf

Question 6. (12 marks)

Marks

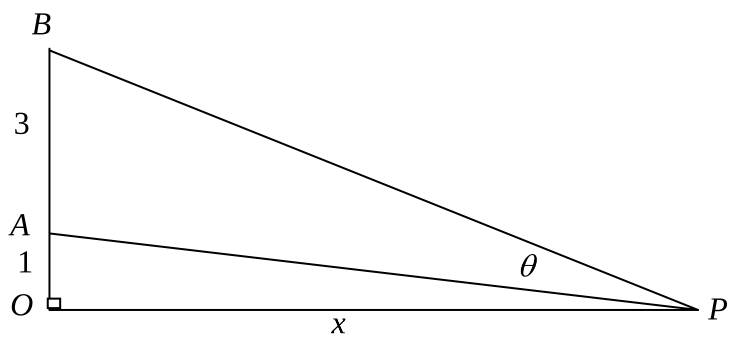
(a) Consider the functions $y = 1 + \cos\theta$ and $y = \sin^2\theta$. **6**

(i) Sketch the graphs of the curves, on the same axes, in the domain $-\pi \leq \theta \leq \pi$.

(ii) Show algebraically that the curves intersect at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

(iii) Find the area enclosed by the curves between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

(b) **6**



In the diagram, a vertical pole *AB*, 3 metres high, is placed on top of a support 1 metre high. The pole subtends an angle of θ radians at the point *P*, which is *x* metres from the base *O* of the support.

(i) Show that $\theta = \tan^{-1}\frac{4}{x} - \tan^{-1}\frac{1}{x}$.

(ii) Show that θ is a maximum when $x = 2$.

(iii) Deduce that the maximum angle subtended at *P* is $\theta = \tan^{-1}\frac{3}{4}$.

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This is the end of the paper.

2010 Extn(1) Task(3)

QUESTION ONE

a) $LHS = \dfrac{\sin A}{\cos A + \sin A} + \dfrac{\sin A}{\cos A - \sin A}$

$= \dfrac{\sin A(\cos A - \sin A) + \sin A(\cos A + \sin A)}{(\cos A + \sin A)(\cos A - \sin A)}$

$= \dfrac{\sin A\cos A - \sin^2 A + \sin A\cos A + \sin^2 A}{\cos^2 A - \sin^2 A}$

$= \dfrac{2\sin A\cos A}{\cos 2A}$

$= \dfrac{\sin 2A}{\cos 2A}$ 2

$= \tan 2A$

$= RHS$

b) $\lim_{x\to 0} \dfrac{\sin 3x}{3} = \lim_{x\to 0} 3\,\dfrac{\sin 3x}{3}$

$= (3)(1)$

$= 3$ 1

c) $\int_{\frac{1}{2}}^{1} 4x(2x-1)^3 \, dx$

Let $u = 2x - 1 \Rightarrow x = \dfrac{u+1}{2}$

$du = 2 \cdot dx$

when $x = 1/2$ $u = 0$

when $x = 1$ $u = 1$

$\int_{\frac{1}{2}}^{1} 4x(2x-1)^3 \, dx = \int_0^1 2\left(\dfrac{u+1}{2}\right)(u^3) \, du$

$= \int_0^1 u^4 + u^3 \, du$

$= \left[\tfrac{1}{5}u^5 + \tfrac{1}{4}u^4\right]_0^1$

$= \left(\tfrac{1}{5} + \tfrac{1}{4}\right) - 0$

$= 9/20$ 4

(d) $A = \frac{1}{2}r^2\theta = 3\pi/4$

$\theta = \pi/6$

$\frac{1}{2}r^2\left(\frac{\pi}{6}\right) = 3\pi/4$

$\frac{1}{2}r^2 = 9/2$

$r^2 = 9$

$r = 3$

$l = r\theta$

$= 3 \times \pi/6$ 2

$= \dfrac{\pi}{2}$

(e) ~~$\sin\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$~~

~~$\sin\left(\theta - \frac{\pi}{4}\right) = \sin(\pi/4)$~~

~~$\theta - \pi/4 = n\pi + (-1)^n \frac{\pi}{4}$~~

~~$\theta = n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{4}$~~

~~$= \frac{\pi}{4}(4n + (-1)^n + 1)$~~

~~when n is odd~~

~~$\theta = \pi n$~~

~~when n is even.~~

~~$\theta = \pi/2(2n+1)$~~

$\sin(\theta - \pi/4) = \frac{1}{\sqrt{2}}$

$\sin(\theta - \pi/4) = \sin(\pi/4)$

$\theta - \pi/4 = \pi/4,\ 3\pi/4$

$\theta = \pi/2,\ \pi$

in domain $0 < \theta < 2\pi$ 3

2010 Extn(1) Task(3)

QUESTION TWO.

a)(i) $\frac{d}{dx}(x^2 \tan 2x)$

$= uv' + u'v$

$u = x^2$ $\quad v = \tan 2x$ (2)

$u' = 2x$ $\quad v' = 2\sec^2 2x$

$= x^2(2\sec^2 2x) + 2x(\tan 2x)$

$= 2x^2 \sec^2 2x + 2x \tan 2x$

$= 2x(x\sec^2 2x + \tan 2x)$

(ii) $\frac{d}{dx}(\cos^{-1} 3x)$

$= \frac{d}{dx}\left(\cos^{-1} \frac{x}{\frac{1}{3}}\right) = \frac{-1}{\sqrt{(\frac{1}{3})^2 - x^2}}$

$= \frac{-1}{\sqrt{\left(\frac{1-9x^2}{9}\right)}}$

$= \frac{-3}{\sqrt{(1-9x^2)}}$ (2)

(iii) $\frac{d}{dx}(\ln_e(\cos x))$

$= \frac{1}{\cos x}(-\sin)$

$= -\tan x$ (2)

b)(i) $\int \frac{1}{\sqrt{4-x^2}} \cdot dx = \sin^{-1}\frac{x}{2} + C$ (2)

ii) $\int \sin(1-3x) \cdot dx$ (2)

$= \frac{-1}{-3}\cos(1-3x) + C$

$= \frac{1}{3}\cos(1-3x) + C$

iii) $\int \sin x \cos^3 x \, dx = \frac{-\cos^4 x}{4} + C$ (2)

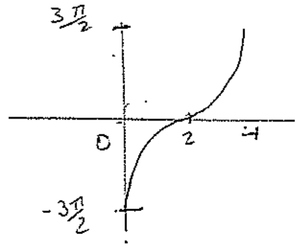
QUESTION 3

a)

(i) Domain: $0 \le x \le 4$

Range $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

(ii)



(iii) $f(1) = 3\sin^{-1}(-\frac{1}{2}) = -\pi/2$

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2 x \csc^2 x \, dx$ $\quad u = \cot x$, $\frac{du}{dx} = -\csc^2 x$

$-\int_0^1 u^2 \csc^2 x \frac{du}{\csc^2 x}$ $\quad x = \frac{\pi}{2}\ u = 0$, $x = \frac{\pi}{4}\ u = 1$

$\left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3} - 0$

$= \frac{1}{3}$

(c) $x^3 - 103x + 102.5 = 0$ root near 1

$a_1 = 1 - \frac{f(1)}{f'(1)}$ $\quad f'(x) = 3x^2 - 103$

$a_1 = 1 - \frac{0.5}{-100}$

$a_1 = 1.005$

(d) $x = 2\cos(t/4)$

$v = \frac{dx}{dt} = -\frac{1}{2}\sin(t/4)$

$a = \frac{dv}{dt} = -\frac{1}{8}\cos(t/4)$

$t = \frac{4\pi}{3}$ $\quad a = -\frac{1}{8}\cos \pi/3$

$a = -\frac{1}{16}$ m/s^2

QUESTION 4

(a) $\ddot{x} = 4(40 - x)$ $\quad t = 0\ x = 0\ v = 0$

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4(40 - x)$

$\frac{1}{2}v^2 = 160x - 2x^2 + C$

$v = 0\ x = 0\ C = 0$

$v^2 = 320x - 4x^2 = 4(80x - x^2)$

(ii) from (i) $v = 0$ when $4(80x - x^2) = 0$

$x = 0, 80$

particle moves between $x = 0$ & $x = 80$

(iii) Max velocity $\ddot{x} = 0$

$v = 2\sqrt{80 \times 40 - 40^2}$

max $= 80$ m/s

4(b)(i) $9y = 8x^2$

$x^2 = \frac{9}{8}y$

$V = \pi\int_0^h \frac{9}{8}y\, dy$

$= \pi\left[\frac{9}{16}y^2\right]_0^h$

$= \frac{9}{16}\pi h^2$ cm^3

(ii) $\frac{dV}{dt} = 20$ $\quad \frac{dV}{dh} = \frac{9}{8}\pi h$ (from (i))

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$= 20 \times \frac{8}{9\pi \times 4}$

$= \frac{40}{9\pi}$ cm/s

≈ 1.415 cm/s

SECTION C

Q5 (a)(i) $5! = \boxed{120.}$ 2

(ii) O E O E O

This is the only possibility with all odd seperated.

$5! - 2! \times 3! = \boxed{108.}$ 2

(b) $\int_0^{\frac{\pi}{8}} \sin 2x \cos^2 2x \, dx.$

let $u = \cos 2x.$

$\frac{du}{dx} = -2\sin 2x$

$-\frac{1}{2} du = \sin 2x \, dx.$

$= -\frac{1}{2}\int_1^{\frac{1}{\sqrt{2}}} u^2 du.$

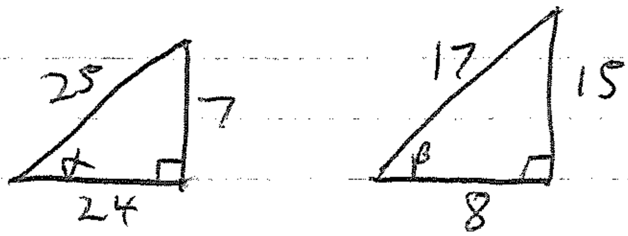
$= -\frac{1}{2}\left[\frac{u^3}{3}\right]_1^{\frac{1}{\sqrt{2}}}$

$= -\frac{1}{6}\left[u^3\right]_1^{\frac{1}{\sqrt{2}}}$

$= -\frac{1}{6}\left(\left(\frac{1}{\sqrt{2}}\right)^3 - 1\right).$

$= \boxed{\frac{1}{6} - \frac{1}{12\sqrt{2}}}$ 2.

(c)



$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ 2

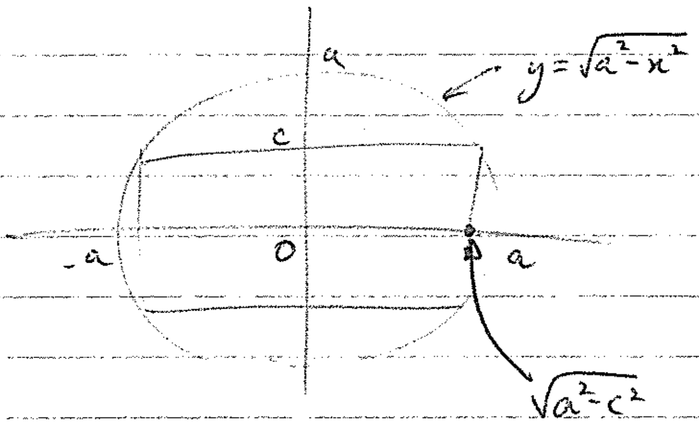
$= \frac{24}{25} \times \frac{8}{17} + \frac{7}{25} \times \frac{15}{17}.$

$= \boxed{\frac{297}{425}}$

sphere radius a

cylindrical hole radius c.

$$\frac{4}{3}\pi\left(a^2-c^2\right)^{3/2}$$



when $y=c$

$$c=\sqrt{a^2-x^2}$$
$$c^2=a^2-x^2$$
$$x^2=a^2-c^2$$
$$x=\pm\sqrt{a^2-c^2}$$

$$V=\pi\int_a^b y^2\,dx$$

4

$$\tfrac{1}{2}V=\pi\int_0^{\sqrt{a^2-c^2}}(a^2-x^2)\,dx-\pi\int_0^{\sqrt{a^2-c^2}}c^2\,dx$$

$$=\pi\left[a^2x-\frac{x^3}{3}\right]_0^{\sqrt{a^2-c^2}}-\pi\left[c^2x\right]_0^{\sqrt{a^2-c^2}}$$

$$=\pi\left[a^2\sqrt{a^2-c^2}-\frac{(a^2-c^2)\sqrt{a^2-c^2}}{3}-0\right\}-\pi\left[c^2\sqrt{a^2-c^2}-0\right]$$

$$=\pi\left[a^2\sqrt{a^2-c^2}-\frac{a^2\sqrt{a^2-c^2}}{3}+\frac{c^2\sqrt{a^2-c^2}}{3}-c^2\sqrt{a^2-c^2}\right]$$

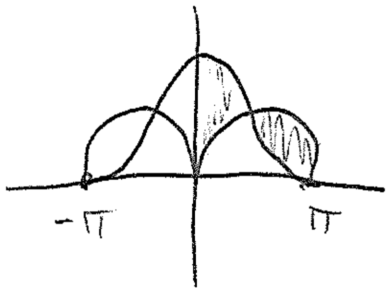
$$=\pi\left[\frac{2a^2\sqrt{a^2-c^2}}{3}-\frac{2c^2\sqrt{a^2-c^2}}{3}\right]$$

$$=\frac{2\pi}{3}\left[\sqrt{a^2-c^2}\left(a^2-c^2\right)\right]$$

$$=\frac{2\pi}{3}\left(a^2-c^2\right)^{3/2}$$

$$V=\frac{4\pi}{3}\left(a^2-c^2\right)^{3/2}$$

Q6 (a)(i) $y = 1+\cos\theta \quad y = \sin^2\theta.$



2

(ii) $1+\cos\theta = 1-\cos^2\theta.$

$\cos^2\theta + \cos\theta = 0.$

$\cos\theta(\cos\theta+1) = 0.$

$\cos\theta = 0$ 2 $\cos\theta = -1$

$\theta = \pi, -\pi.$

$\theta = \frac{\pi}{2}, -\frac{\pi}{2}$

(iii) $2\int_0^{\frac{\pi}{2}} 1+\cos\theta - \sin^2\theta \, d\theta + \cancel{2\int_{\frac{\pi}{2}}^{\pi} \sin^2\theta - 1 - \cos\theta \, d\theta}.$

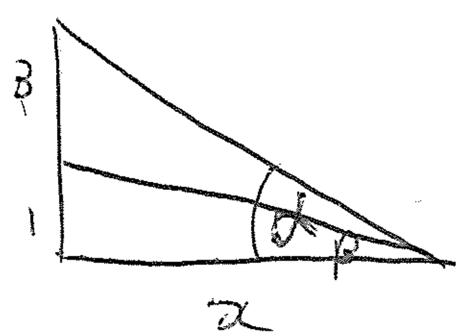
$= \int_0^{\frac{\pi}{2}} 2+2\cos\theta - 1 + \cos 2\theta \, d\theta + \cancel{\int_{\frac{\pi}{2}}^{\pi} -1 - \cos 2\theta - 2\cos\theta \, d\theta}.$

$= \left[\theta + 2\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} + \cancel{\left[-\theta - \frac{1}{2}\sin 2\theta - 2\sin\theta\right]_{\frac{\pi}{2}}^{\pi}}$

$= \left(\frac{\pi}{2}+2+0\right) - (0+0+0) + \cancel{(-\pi - 0 - 0) - \left(-\frac{\pi}{2} - 0 - 2\right)}$

$= \frac{\pi}{2} + 2$ 2

(b)



$\theta = \alpha - \beta.$

(i) $\tan\alpha = \frac{4}{x} \qquad \tan\beta = \frac{1}{x}.$

2

$\theta = \tan^{-1}\frac{4}{x} - \tan^{-1}\frac{1}{x}$

(ii). $\theta' = \frac{1}{1+(\frac{4}{x})^2} \times -\frac{4}{x^2} - \frac{1}{1+(\frac{1}{x})^2} \times -\frac{1}{x^2}.$

$= \frac{1}{x^2+1} - \frac{4}{x^2+16}$

$\theta'(x) = \frac{12 - 3x^2}{(x^2+1)(x^2+16)}$

3

$\theta'(x) = 0.$ when $12 - 3x^2 = 0.$

$x = \pm 2.$

$\theta'(1) = \frac{12-3}{(2)(17)} > 0 \qquad \theta'(3) = \frac{12-27}{(10)(25)} < 0$

$\therefore$ $x = 2$ is a maximum.

(iii) $\theta(2) = \tan^{-1}(2) - \tan^{-1}(\frac{1}{2}).$

$\tan(\theta(2)) = \tan(\tan^{-1}2 - \tan^{-1}\frac{1}{2}).$

1

$= \frac{2 - \frac{1}{2}}{1 + 1^2}$

$= \frac{\frac{3}{2}}{2} = \frac{3}{4}.$ i.e. $\theta = \tan^{-1}(\frac{3}{4}).$