SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

June 2011
Assessment Task 3
Year 12

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- All necessary working should be shown in every question.
- All answers to be given in simplified exact form unless otherwise stated.

Total Marks – 92

- Attempt questions 1-6
- All questions are of equal value.
- Start each new question in a separate answer booklet.
- Hand in your answers in 4 separate bundles: Section A (Questions 1 and 2), Section B (Questions 3 and 4, Section C (Questions 5 and 6)

Examiner: *A Ward*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

Start a new booklet.

Section A

Question 1 (15 marks). **Marks**

a) In how many ways can 8 people sit at a round table? 1

b) For the polynomial $(a-6)x^5+(2b+1)x^2+(12-4c)$, state the possible value(s) of *a*, *b* and *c* if the polynomial is: 5

(i) monic of degree 5.

(ii) of degree 2.

(iii) a zero polynomial.

c) Find $\frac{dy}{dx}$ given the following: 9

(i) $y=\frac{1}{x+3}$

(ii) $y=\sin\left(1+x^3\right)$

(iii) $y=e^{4x+2}$

(iv) $y=\frac{\ln(x+2)}{\ln\left(x^2+1\right)}$

End of Question 1

Question 2 (15 Marks). **Marks**

a) (i) Given $f(x)=\dfrac{5-3x}{4}$, find $f^{-1}(x)$ 3

(ii) Sketch the graphs $f(x)$ and $f^{-1}(x)$ on the same axes.

b) Find the following: 4

(i) $\displaystyle\int \frac{6}{3x+1}\,dx$

(ii) $\displaystyle\int \frac{e^x+1}{e^x}\,dx$

c) Evaluate: $\displaystyle\int_1^2 \frac{3x^2-5x+7}{x}\,dx$ 3

d) Find an exact value for $\sin 15^\circ$, showing all working. 3

e) If $y=\sin^{-1}(x)$, find $\dfrac{d^2y}{dx^2}$. 2

End of Question 2

End of Section A

Section B – Start a new booklet.

Question 3 (15 Marks). **Marks**

a) $Q(x) = ax^2 + bx + c$ and when $Q(x)$ is divided by $(x - m)$ or $(x - n)$, the remainders are the same. Prove that, if $m \neq n$ then $(m + n)$ is equal to the sum of the roots of $Q(x)$. 4

b) (i) What are the range and period of $y = 2\cos\left(\frac{x}{2}\right)$? 5

(ii) Draw the graph of $y = 1 - 2\cos\left(\frac{x}{2}\right)$ for $-2\pi \leq x \leq 2\pi$.

c) (i) In how many ways can 10 individuals be divided into 2 groups of 5? 6

(ii) If the two youngest individuals must be in the same group, how many ways are there now?

(iii) How many ways are there, if the two youngest individuals must not be in the same group?

End of Question 3

Question 4 (15 Marks). **Marks**

a) (i) Express $\sqrt{3}\cos 2t - \sin 2t$ in the form $A\cos\left(2t+\alpha\right)$ with $A > 0$ and $0 \le \alpha \le \frac{\pi}{2}$. 4

(ii) Find, in exact form, the general solution to $\sqrt{3}\cos 2t - \sin 2t = 1$

b) Show that: 3

$$\frac{n!}{\left(n-r\right)!r!} + \frac{n!}{\left(n-r+1\right)!\left(r-1\right)!} = \frac{\left(n+1\right)!}{r!\left(n-r+1\right)!}$$

c) For $n = 1, 2$ and 3, find the value of : 4

$$\int_{\frac{\pi}{2}}^{\pi} \cos nx \, dx$$

d) (i) Find the second derivative of $y = \ln\left(x^2+4\right)$ 4

(ii) .Write the equation of the tangent to $y = x\tan^{-1}\left(\frac{x}{2}\right)$ at x=2

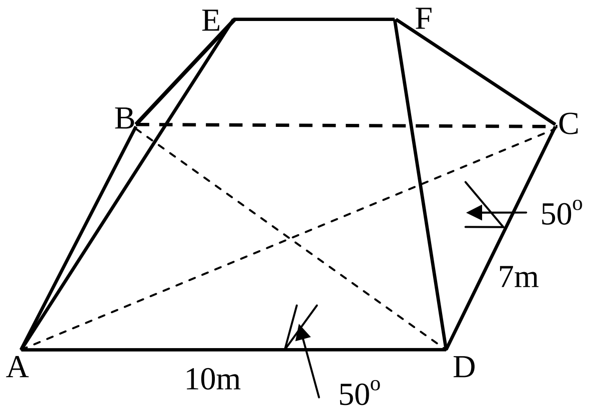
End of Question 4

End of Section B

Section C – Start a new booklet.

Question 5 (15 Marks). **Marks**

a)



A roof of a house has rectangular base 7m by 10m. Each of the four sloping faces makes an angle of 50^{o} with the horizontal. 5

(i) Calculate the length of the ridge EF

(ii) Calculate the total surface area of the sloping faces to 1 decimal place.

b) Prove : 4

$$\sec 2A + \tan 2A \equiv \frac{\cos A + \sin A}{\cos A - \sin A}$$

c) Use the substitution $u = 2\sin x$ to find: 3

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\cos x}{1 + 4\sin^2 x}\,dx$$

d) Factorise $24x^3 - 14x^2 - 63x + 45$, given that one of the zeros is $\frac{3}{2}$. 3

End of Question 5

Question 6 (17 marks). **Marks**

a) Solve the following for $0 \le t \le 2\pi$: 2

$$\sin 2t + 4\sin t + 2\cos t = -4$$

b) If $t = \tan\left(\frac{\theta}{2}\right)$, express in terms of t: $\dfrac{1-2\sin\theta}{2\cos\theta+1}$ 3

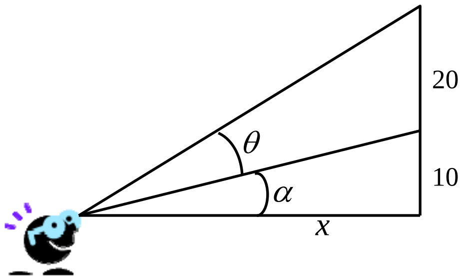
Leave your answer in simplified exact form.

c) (i) Find the coordinates of P, the point of intersection of the curves: 5

$$y = e^{x}$$
$$y = 2 + 3e^{-x}$$

(ii) If these curves cut the y-axis at the points A and B. Calculate the area bounded by AB and the arcs AP and BP.

d) A movie screen on a wall is 20 metres high and 10 metres above eye level. You are x metres from the screen. 7



(i) Show that $\theta = \tan^{-1}\left(\frac{30}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)$

(ii) Find $\lim_{x\to 0}\left(\tan^{-1}\left(\frac{30}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)\right)$

(iii) Find $\lim_{x\to \infty}\left(\tan^{-1}\left(\frac{30}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)\right)$

(iv) At what distance should you position yourself so that the viewing angle θ of the movie screen is as large as possible?

End of Question 6.

End of Section C.

End of Examination.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

1 (a) $7!$ 1

(b) (i) $(a-6) = 1$ 1

$a = 7$

(ii) $a - 6 = 0$ 1

$a = 6$

(iii) $a = 6,\ b = -\frac{1}{2}\quad c = 3$ 3

(c) (i) $y = (x+3)^{-1}$

$\frac{dy}{dx} = -(x+3)^{-2}\cdot +1$ 2

$= \frac{-1}{(x+3)^2}$

(ii) $y = \sin(1+x^3)$

$\frac{dy}{dx} = \cos(1+x^3)\cdot 3x^2$ 2

$= 3x^2\cos(1+x^3)$

(iii) $y = e^{4x+2}$ 2

$\frac{dy}{dx} = e^{4x+2}\cdot 4$

$= 4e^{4x+2}$

(iv) $y = \frac{\ln(x+2)}{\ln(x^2+1)}$ 3

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{\ln(x^2+1)\frac{1}{x+2} - \ln(x+2)\frac{2x}{x^2+1}}{[\ln(x^2+1)]^2}$

$= \frac{\frac{\ln(x^2+1)}{x+2} - \frac{2x\ln(x+2)}{x^2+1}}{(\ln(x^2+1))^2}$

$= \frac{(x^2+1)\ln(x^2+1) - 2x(x+2)\ln(x+2)}{(x+2)(x^2+1)[\ln(x^2+1)]^2}$

15

2 (a) $f(x) = \frac{5-3x}{4}$; $x \in R$, $y \in R$

$y = \frac{5-3x}{4}$

$y = \frac{5}{4} - \frac{3}{4}x$ 2

For $f^{-1}(x)$

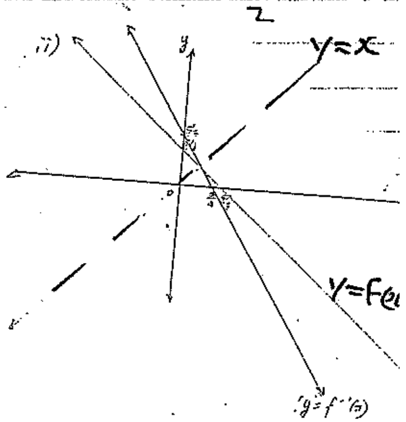
$x = \frac{5}{4} - \frac{3}{4}y$

$4x = 5 - 3y$

$3y = 5 - 4x$

$y = \frac{5}{3} - \frac{4}{3}x$

$f^{-1}(x) = \frac{5}{3} - \frac{4}{3}x$



(b) (i) $\int \frac{6}{3x+1}\,dx$

$= 2\int \frac{3}{3x+1}\,dx$ 2

$= 2\ln(3x+1) + c$

(ii) $\int \frac{e^x+1}{e^x}\,dx$

$= \int 1 + e^{-x}\,dx$ 2

$= x - e^{-x} + c$

(c) $\int_1^2 \frac{3x^2 - 5x + 7}{x}\,dx$

$= \int_1^2 3x - 5 + \frac{7}{x}\,dx$ 1

$= \left[\frac{3x^2}{2} - 5x + 7\ln x\right]_1^2$ 1

$= (6 - 10 + 7\ln 2) - (\frac{3}{2} - 5 + 7\ln 1)$

$= 7\ln 2 - 4 + 3\frac{1}{2}$ 1

$= 7\ln 2 - \frac{1}{2}$

(d)

$$y = \sin 15^\circ$$

$$y = \sin(45-30)^\circ \quad (1)$$

$$= \sin 45 \cos 30 - \cos 45 \sin 30$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \quad (2)$$

(e)

$$y = \sin^{-1}(x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$= (1-x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}} \times (-2x) \quad (1)$$

$$= \frac{x}{\sqrt{(1-x^2)^3}} \quad (1)$$

15

QUESTION 3

(a) $Q(m) = Q(n)$

$am^2 + bm + c = an^2 + bn + c$

$a(m^2 - n^2) = -b(m-n)$

$a(m+n) = -b$

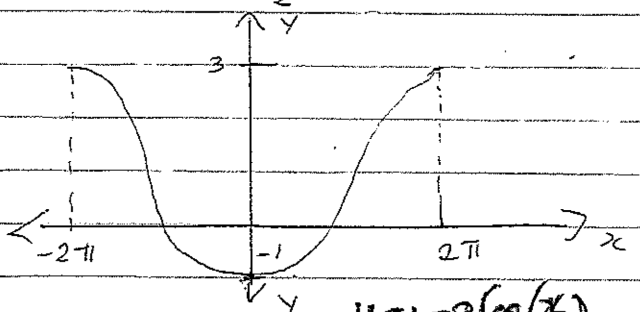
$m+n = \frac{-b}{a}$

$=$ Sum of roots

(b)

Range $-2 \le y \le 2$

Period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$



$y = 1 - 2\cos\left(\frac{x}{2}\right)$

(c) (i) $^{10}C_5 \div 2 = 126$

(ii) $^{8}C_3 = 56$

(iii) $^{8}C_4 = 70$

or (i) − (ii)

4(a)(ii)

$2\cos(2t + \pi/6) = 1$

$\cos(2t + \pi/6) = \frac{1}{2}$

$2t + \pi/6 = \pm \pi/3 + 2n\pi$

$2T = -\pi/6 \pm \pi/3 + 2n\pi$

$t = n\pi - \pi/12 \pm \pi/6$

QUESTION 4

(a) $\sqrt{3}\cos t - \sin 2t = A\cos(2t + \alpha)$

$= A(\cos 2t \cos\alpha - \sin 2t \sin\alpha)$

$A\cos\alpha = \sqrt{3}$

$A\sin\alpha = 1$

$A = \sqrt{3+1} = 2$

$\cos A = \sqrt{3}/2 \quad \sin A = \frac{1}{2} \quad \alpha = \pi/6$

$\sqrt{3}\cos t - \sin 2t = 2\cos(2t + \pi/6)$

(b) $\frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!} = \frac{n!(n-r+1) + n!(r)}{(n-r+1)!\,r!}$

$= \frac{n!(n-r+1-r)}{(n-r+1)!\,r!} = \frac{n+1!}{(n-r+1)!\,r!}$

(c) $n=1 \quad \sin\pi - \sin\frac{\pi}{2} = 0 - 1 = -1$

$n=2 \quad \frac{1}{2}\sin 2\pi - \frac{1}{2}\sin\pi = 0 - 0 = 0$

$n=3 \quad \frac{1}{3}\sin 3\pi - \frac{1}{3}\sin\frac{3\pi}{2} = 0 - -\frac{1}{3} = \frac{1}{3}$

(d) (i) $y = \ln(x^2+4)$

$y' = \frac{2x}{x^2+4}$

$y'' = \frac{(x^2+4)2 - 2x \times 2x}{(x^2+4)^2}$

$= \frac{-2x^2+8}{(x^2+4)^2}$

(ii) $y = x\tan^{-1}\frac{x}{2}$

$y' = \tan^{-1}\frac{x}{2} \times 1 + x \times \frac{2}{4+x^2}$

$= \tan^{-1}\frac{x}{2} + \frac{2x}{4+x^2}$

$x = 2 \quad m = \frac{\pi}{4} + \frac{1}{2} \quad y = \frac{\pi}{2}$

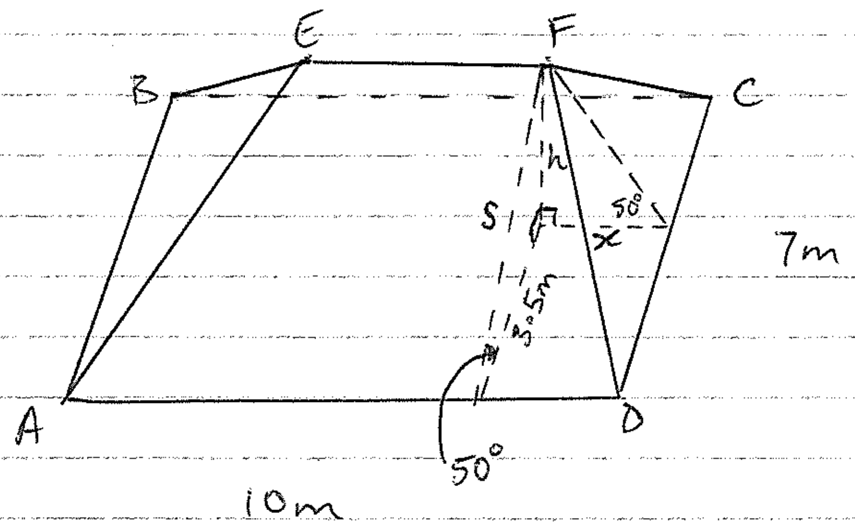
eqn $y - \frac{\pi}{2} = \left(\frac{\pi}{4} + \frac{1}{2}\right)(x - 2)$

$y = x\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{\pi}{2} - 1 + \frac{\pi}{2}$

$y = x\left(\frac{\pi}{4} + \frac{1}{2}\right) - 1$

Question 5

a)



i) $\tan 50° = \frac{h}{3.5}$

$h = 3.5 \tan 50°$

$\tan 50° = \frac{h}{x}$

$x = \frac{3.5 \tan 50°}{\tan 50°}$

$x = 3.5$

$EF = 10 - 2(3.5)$

$= 3 \text{ m}$

ii) $\cos 50° = \frac{3.5}{S}$

$S = \frac{3.5}{\cos 50°}$

$S.A = 2 \times \left[\frac{1}{2} \times 7 \times \frac{3.5}{\cos 50°}\right] + 2 \times \left[\frac{3+10}{2} \times \frac{3.5}{\cos 50°}\right]$

$= 108.9 \text{ m}^2$ (to 1 decimal place)

b) $LHS = \sec 2A + \tan 2A$

$$= \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A}$$

$$= \frac{1 + \sin 2A}{\cos 2A}$$

$$= \frac{1 + 2\sin A\cos A}{\cos^2 A - \sin^2 A}$$

$$= \frac{\sin^2 A + \cos^2 A + 2\sin A\cos A}{\cos^2 A - \sin^2 A}$$

$$= \frac{(\cos A + \sin A)^2}{(\cos A - \sin A)(\cos A + \sin A)}$$

$$= \frac{\cos A + \sin A}{\cos A - \sin A}$$

$= RHS$

c) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\cos x}{1 + 4\sin^2 x}\,dx$$

$u = 2\sin x$

$\frac{du}{dx} = 2\cos x$

$dx = \frac{du}{2\cos x}$

unit change

$x = \frac{\pi}{6}$, $u = 2\sin\frac{\pi}{6}$, $u = 1$

$x = \frac{\pi}{3}$, $u = 2\sin\frac{\pi}{3}$, $u = \sqrt{3}$

$$= \int_1^{\sqrt{3}} \frac{8\cancel{\cos x}}{1 + u^2} \cdot \frac{du}{2\cancel{\cos x}}$$

$$= 4\int_1^{\sqrt{3}} \frac{du}{1 + u^2}$$

$$= 4\left[\tan^{-1} u\right]_1^{\sqrt{3}}$$

$$= 4\left[\tan^{-1}\sqrt{3} - \tan^{-1} 1\right]$$

$$= 4\left[\frac{\pi}{3} - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{3}$$

d) $P(x) = 24x^3 - 14x^2 - 63x + 45$

If $\frac{3}{2}$ is a zero, $(2x-3)$ is a factor of $P(x)$.

$$\begin{array}{r} 12x^2 + 11x - 15 \\ 2x-3 \overline{\big) 24x^3 - 14x^2 - 63x + 45} \\ \underline{24x^3 - 36x^2} \qquad\qquad\quad \\ 22x^2 - 63x \qquad \\ \underline{22x^2 - 33x} \qquad \\ -30x + 45 \\ \underline{-30x + 45} \\ 0 \end{array}$$

$P(x) = (2x-3)(12x^2 + 11x - 15)$

$\times$ | -180
$+$ | 11
$20, -9$

$= (2x-3)\dfrac{(12x+20)(12x-9)}{\underset{4\times 3}{12}}$

$= (2x-3)(3x+5)(4x-3)$

Question 6

a) $\sin 2t + 4\sin t + 2\cos t = -4$

$\underbrace{2\sin t\cos t + 4\sin t} + \underbrace{2\cos t + 4} = 0$

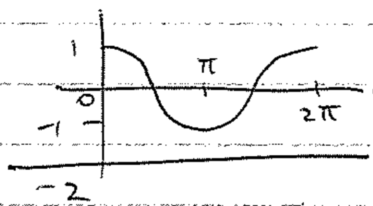
$2\sin t(\cos t + 2) + 2(\cos t + 2) = 0$

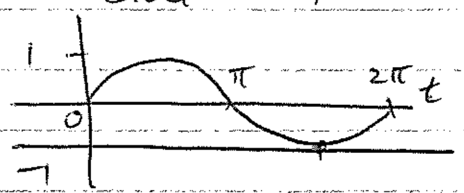
$(\cos t + 2)(2\sin t + 2) = 0$

$\cos t + 2 = 0$ $\qquad$ $2\sin t + 2 = 0$

$\cos t = -2$ $\qquad$ $2\sin t = -2$

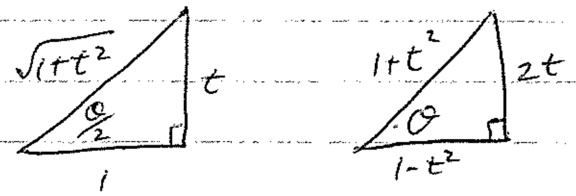
$\sin t = -1$





No solution $\qquad$ $t = \dfrac{3\pi}{2}$

b) $t = \tan\left(\frac{\theta}{2}\right)$



$$\frac{1-2\sin\theta}{2\cos\theta+1} = \frac{1-2\left(\frac{2t}{1+t^2}\right)}{2\left(\frac{1-t^2}{1+t^2}\right)+1} \times \frac{1+t^2}{1+t^2}$$

$$= \frac{1+t^2-4t}{2-2t^2+1+t^2}$$

$$= \frac{t^2-4t+1}{3-t^2}$$

c) i) $y = e^x$ ——— ①

$y = 2+3e^{-x}$ ——— ②

sub. ① into ②

$e^x = 2+3e^{-x}$

multiply both sides by e^x

$(e^x)^2 = 2e^x + 3$

$(e^x)^2 - 2e^x - 3 = 0$ $\quad$ × −3, + −2, −3, 1

$(e^x-3)(e^x+1) = 0$

$e^x = 3$ $\quad$ or $\quad$ $e^x = -1$

$x = \ln 3$ $\quad$ No solution since $e^x > 0$.

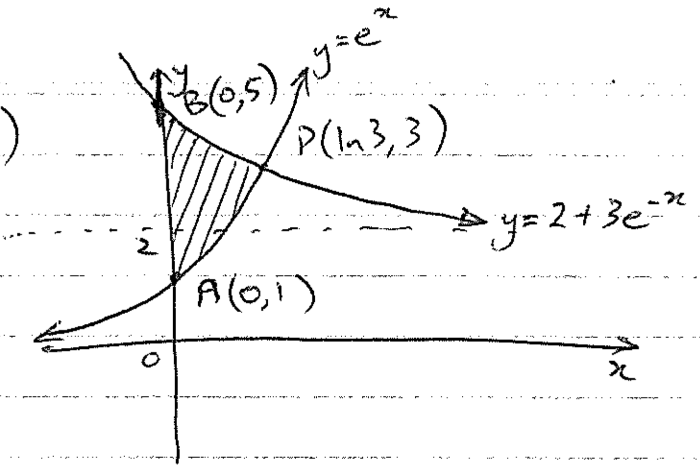
sub. into ①

$y = e^{\ln 3}$

$y = 3$

$\therefore$ P has coordinates $(\ln 3, 3)$

ii)



$$A = \int_0^{\ln 3} \left(2 + 3e^{-x} - e^{x}\right) dx$$

$$= \left[2x - 3e^{-x} - e^{x} \right]_0^{\ln 3}$$

$$= 2\ln 3 - 3e^{-\ln 3} - e^{\ln 3} - \left(2(0) - 3e^{-(0)} - e^{0}\right)$$

$$= 2\ln 3 - 3e^{\ln(\frac{1}{3})} - 3 - (-3 - 1)$$

$$= 2\ln 3 - 3(\tfrac{1}{3}) + 1$$

$$= 2\ln 3 \text{ units}^2.$$

d) i) $\tan(\theta + \alpha) = \frac{30}{x}$

$$\theta + \alpha = \tan^{-1}\left(\frac{30}{x}\right)$$

$$\tan(\alpha) = \frac{10}{x}$$

$$\alpha = \tan^{-1}\left(\frac{10}{x}\right)$$

$$\theta = (\theta + \alpha) - \alpha$$

$$= \tan^{-1}\left(\frac{30}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)$$

ii) as $x \to 0$

$\frac{30}{x} \to \infty$

$\tan^{-1}\left(\frac{30}{x}\right) \to \frac{\pi}{2}$

similarly, $\tan^{-1}\left(\frac{10}{x}\right) \to \frac{\pi}{2}$

$\therefore \lim_{x \to 0} \left(\tan^{-1}\left(\frac{30}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)\right)$

$= \frac{\pi}{2} - \frac{\pi}{2}$

$= 0$

iii) as $x \to \infty$

$\frac{30}{x} \to 0$

$\tan^{-1}\left(\frac{30}{x}\right) \to 0$

similarly, $\tan^{-1}\left(\frac{10}{x}\right) \to 0$

$\therefore \lim_{x \to \infty} \left(\tan^{-1}\left(\frac{30}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)\right)$

$= 0 - 0$

$= 0$

iv) $\theta = \tan^{-1}\left(\frac{30}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)$

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{30}{x}\right)^2} \cdot \frac{-30}{x^2} - \frac{1}{1+\left(\frac{10}{x}\right)^2} \cdot \frac{-10}{x^2}$$

$$= \frac{-30}{x^2+900} + \frac{10}{x^2+100}$$

$$= \frac{-30(x^2+100) + 10(x^2+900)}{(x^2+900)(x^2+100)}$$

For stationary points let $\frac{d\theta}{dx} = 0$.

$$-30x^2 - 3000 + 10x^2 + 9000 = 0$$
$$-20x^2 + 6000 = 0$$
$$20x^2 = 6000$$
$$x^2 = 300$$
$$x = \pm\sqrt{300}$$
$$x = 10\sqrt{3}\ m \quad \text{since } x \text{ is a distance}$$

θ is continuous for $x > 0$.
From the limits in (ii) & (iii)
θ is a maximum when $x = 10\sqrt{3}\,m$.