SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

June 2012

Assessment Task 3
Year 12

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question if full marks are to be awarded.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 62

- Attempt sections A – C.
- Start each **NEW** section in a separate answer booklet.
- Hand in your answers in 3 separate bundles:
  Section A
  Section B
  Section C

Examiner: *J. Chen*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

START A NEW ANSWER BOOKLET

SECTION A [20 marks] **Marks**

For these 10 questions there is one correct answer per question. Write down in your answer booklet the question number and letter of your answer.

1. 1.58 radians approximately equal **[1]**

(A) $90°31'$

(B) $90°32'$

(C) $90°53'$

(D) None of the above

2. **[1]**

$$\lim_{x \to 0} \frac{\tan 3x}{x}$$

equals

(A) $\frac{1}{3}$

(B) 3

(C) 0

(D) 1

3. **[1]**

$$\int \frac{dx}{x^2 + 4x + 5}$$

equals

(A) $\tan^{-1}\left(\frac{x+2}{2}\right) + C$

(B) $\tan^{-1}\left(\frac{x}{2}\right) + C$

(C) $\tan^{-1}(x + 2) + C$

(D) $2\tan^{-1}\left(\frac{x}{2}\right) + C$

4. **[1]**

$$\int_0^{\frac{\pi}{4}} (1 + \tan^2 x)\,.\,dx$$

equals

(A) -1

(B) 0

(C) 1

(D) 2

5. The domain of the function **[1]**

$$y = 2\sin^{-1}\left(\frac{x}{2}\right)$$

is

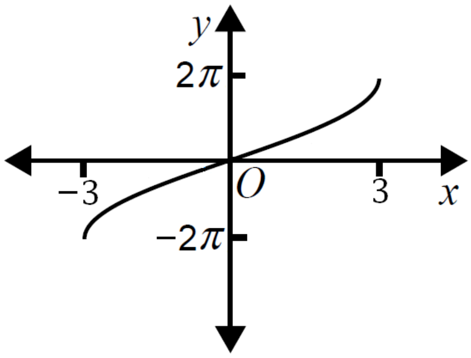
(A) $-2 \le x \le 2$

(B) $-\frac{1}{2} \le x \le \frac{1}{2}$

(C) $-2 \le y \le 2$

(D) $-\frac{1}{2} \le y \le \frac{1}{2}$

6. The diagram below shows the graph of a function. **[1]**



A possible equation for the function is

(A) $y = 4\sin^{-1}(3x)$

(B) $y = 4\sin^{-1}\left(\frac{x}{3}\right)$

(C) $y = \frac{1}{4}\sin^{-1}(3x)$

(D) $y = \frac{1}{4}\sin^{-1}\left(\frac{x}{3}\right)$

7. **[1]**

$$\int_{-2}^{2} \sqrt{4 - x^2}\,.dx$$

equals

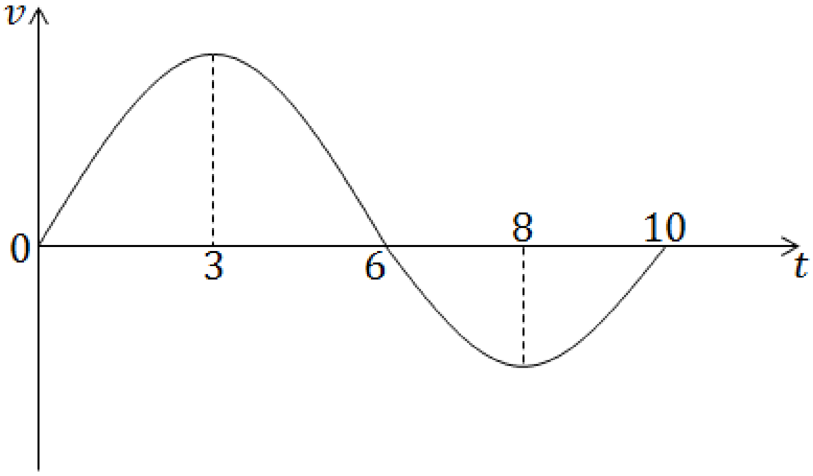
(A) 0

(B) 2π

(C) 4π

(D) $\frac{\pi}{2}$

8. The graph represents the velocity v m/s of a particle after t seconds travelling in a straight line. The particle starts from rest at the origin. **[1]**



When does the particle return to the origin?

(A) $t = 2$

(B) $t = 4$

(C) $t = 10$

(D) The particle does not return to the origin.

9. **[1]**

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right)$$

Equals

(A) $\frac{x\cos x+\sin x}{x^2}$

(B) $\frac{x\cos x-\sin x}{x^2}$

(C) $\frac{\sin x-x\cos x}{x^2}$

(D) $\frac{\sin x+x\cos x}{x^2}$

10. How many solutions does the equation $\sin 2x = \cos x$ have in the domain $0 \leq x \leq 2\pi$? **[1]**

(A) 0

(B) 1

(C) 2

(D) None of the above

End of Multiple Choice Section

11. Find the inverse function f^{-1} of the function f, defined by $f(x) = 2\log_e x + 3$. **[1]**
Express the result in the form y in terms of x.

12. Find the exact value of **[3]**

$$\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5}$$

13. State the domain and range of the function: **[2]**

$$f(x) = 2\cos^{-1}\frac{x}{3}$$

14. **[4]**

(i) Differentiate $x\sin^{-1}x + \sqrt{1-x^2}$.

(ii) Hence evaluate

$$\int_0^1 \sin^{-1}x \,.\, dx$$

End of Section A

START A NEW ANSWER BOOKLET

SECTION B [20 marks] **Marks**

1. Find $\int x\sqrt{1+x^2}\,.\,dx$, using the substitution $u = 1 + x^2$. **[3]**

2. Find **[3]**

$$\int_0^{\pi} \sin^2 x\,.\,dx$$

3. Evaluate **[2]**

$$\lim_{x\to 0}\frac{\tan 6x}{\sin 4x}$$

4. **[4]**

(i) Show that there is a root to the equation $\sin x = x - \frac{1}{2}$ between $x = 0.5$ and $x = 1.8$.

(ii) Taking $x = 1.2$ as a first approximation to this solution, apply Newton's method once to find a closer approximation to the solution. Give your answer correct to two decimal places.

5. **[4]**

(i) Write down a primitive function of $e^{f(x)}.f'(x)$.

(ii) Hence, evaluate

$$\int_{-1}^{1}\frac{2e^{\cos^{-1}x}}{\sqrt{1-x^2}}.\,dx$$

6. The surface area of a cube is increasing at a rate of 12.6 cm^2 per second and initially each edge is 10 cm. **[4]**

(i) If it maintains its cubic shape, find an expression in terms of t for the surface area of the cube after t seconds.

(ii) Hence, or otherwise, find the rate at which the volume of the cube is increasing after 10 seconds.

End of Section B

START A NEW ANSWER BOOKLET

SECTION C [22 marks] **Marks**

1. **[5]**

(i) Prove that

$$\frac{1}{x^2+1} - \frac{1}{x^2+3} = \frac{2}{(x^2+1)(x^2+3)}$$

(ii) Hence determine the value of

$$\int_{-\sqrt{3}}^{1} \frac{dx}{(x^2+1)(x^2+3)}$$

2. **[5]**

(i) Prove by Mathematical Induction that if x is a positive integer then $(x+1)^n - 1$ is divisible by x for all positive integers $n > 0$.

(ii) Hence, deduce that $15^n - 5^n - 3^n + 1$ is divisible by 8 for all positive integers $n > 0$.

3. A particle moves with acceleration $\ddot{x} = 2x^3 - 10x$ where x is the displacement in metres. Initially $v = 0$ when $x = -1$. **[6]**

(i) Find v in terms of x.

(ii) Explain why there is a restriction on x and clearly indicate the restriction.

(iii) Describe briefly what would have happened if the initial conditions where $v = 0$ when $x = 0$.

4. Assume that tides rise and fall in **Simple Harmonic Motion**. On June 25 the depth of water in a harbour entrance at low tide is 2 metres at 11 a.m. At the following high tide at 5:20 p.m. the depth is 6 metres. Calculate the first time period during which a yacht can safely enter the harbour if minimum depth of 3.5 metres of water is needed. **[6]**

End of Section C
End of Exam

1 (B) $90° 32'$

2 $\lim_{x \to 0} \frac{\tan 3x}{x}$

$= \lim_{3x \to 0} \frac{\tan 3x}{3x} \times 3$

$= 1 \times 3$

$= 3$ (B)

3. $\int \frac{dx}{x^2+4x+5}$

$= \int \frac{dx}{(x+2)^2+1}$

$= \tan^{-1}(x+2) + c$ (C)

4 $\int_0^{\pi/4} (1+\tan^2 x)\,dx$

$= \int_0^{\pi/4} \sec^2 x \, dx$

$= [\tan x]_0^{\pi/4}$

$= \tan \pi/4 - \tan 0$

$= 1$ (C)

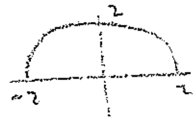
5. $y = 2\sin^{-1}\left(\frac{x}{2}\right)$

$-1 \le \frac{x}{2} \le 1$

$-2 \le x \le 2$ (A)

6. $y = 4\sin^{-1}\frac{x}{3}$ (B)

7. $\int_{-2}^{2} \sqrt{4-x^2}\,dx$



$= \frac{1}{2} \times \pi \times 2^2$

$= 2\pi$ (B)

8. (D) Particle does not return to origin

9. $\frac{d}{dx}\left(\frac{\sin x}{x}\right)$

$= \frac{x \cdot \cos x - \sin x \cdot 1}{x^2}$

$= \frac{x\cos x - \sin x}{x^2}$ (B)

10. $\sin 2x = \cos x$

$\therefore 2\sin x \cos x = \cos x$

$\therefore \cos x(2\sin x - 1) = 0$

$\therefore \cos x = 0$ or $\sin x = \frac{1}{2}$

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

4 solutions (D)

(None of the above)

11. $x = 2\log_e y + 3$

$x - 3 = 2\ln y$

$\ln y = \frac{x-3}{2}$

$y = e^{\frac{x-3}{2}}$ (1)

12. $x = \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5}$

$$\tan x = \tan\left(\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5}\right)$$

$$= \frac{\tan\left(\tan^{-1}\frac{1}{4}\right) + \tan\left(\tan^{-1}\frac{3}{5}\right)}{1 - \tan\left(\tan^{-1}\frac{1}{2}\right)\tan\left(\tan^{-1}\frac{3}{5}\right)}$$

$$= \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4} \times \frac{3}{5}}$$

$$= \frac{\left(\frac{17}{20}\right)}{\left(\frac{17}{20}\right)}$$

$$= 1$$ (3)

$$\therefore x = \frac{\pi}{4}$$

13. $f(x) = 2\cos^{-1}\frac{x}{3}$

Domain $-1 \le \frac{x}{3} \le 1$ (1)

$\therefore -3 \le x \le 3$

Range $0 \le \cos^{-1}\frac{x}{3} \le \pi$

$0 \le 2\cos^{-1}\frac{x}{3} \le 2\pi$ (1)

$\therefore 0 \le y \le 2\pi$

14. (i) $\frac{d}{dx}\left(x\sin^{-1}x + \sqrt{1-x^2}\right)$

$= \sin^{-1}x \cdot 1 + x \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} \times -2x$ (2)

$= \sin^{-1}x$

(ii) $\int_0^1 \sin^{-1}x \, dx = \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^1$

$= \left[1 . \sin^{-1}1 + 0\right] - \left[0 + 1\right]$ (2)

$= \frac{\pi}{2} - 1$

2012 Extension 1 Mathematics Task 3:
Solutions— Section B

1. Find $\int x\sqrt{1+x^2}.dx$, using the substitution $u = 1 + x^2$. [3]

> **Solution:** $\dfrac{du}{dx} = 2x,$
> $$\begin{aligned} \text{I} &= \tfrac{1}{2}\int \frac{du}{\cancel{dx}}.u^{\frac{1}{2}}.\cancel{dx}, \\ &= \tfrac{1}{\cancel{2}}.u^{\frac{3}{2}}.\tfrac{\cancel{2}}{3} + c, \\ &= \frac{(1+x^2)\sqrt{1+x^2}}{3} + c. \end{aligned}$$

2. Find [3]

$$\int_0^{\pi} \sin^2 x.dx$$

> **Solution:**
> $$\begin{aligned} \text{I} &= \tfrac{1}{2}\int_0^{\pi} (1 - \cos 2x)dx, \\ &= \tfrac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}, \\ &= \tfrac{1}{2}\{\pi - 0 - (0 - 0)\}, \\ &= \frac{\pi}{2}. \end{aligned}$$
> $$\begin{aligned} \cos 2x &= 1 - 2\sin^2 x, \\ \sin^2 x &= \frac{1 - \cos 2x}{2}. \end{aligned}$$

3. Evaluate [2]

$$\lim_{x \to 0} \frac{\tan 6x}{\sin 4x}$$

> **Solution:** First method—
> $$\begin{aligned} \lim_{x \to 0} \frac{\tan 6x}{\sin 4x} &= \lim_{x \to 0} \frac{\tan 6x}{6x}.\frac{4x}{\sin 4x}.\frac{6}{4}, \\ &= 1 \times 1 \times \frac{6}{4}, \\ &= \frac{3}{2}. \end{aligned}$$

Solution: Second method—

$$\begin{aligned}
\frac{\tan 6x}{\sin 4x} &= \frac{\sin 6x}{\cos 6x.\sin 4x}, \\
&= \frac{\sin 4x.\cos 2x + \cos 4x.\sin 2x}{\cos 6x.\sin 4x}, \\
&= \frac{\sin 4x.\cos 2x}{\cos 6x.\sin 4x} + \frac{\cos 4x.\sin 2x}{\cos 6x \times 2\sin 2x.\cos 2x}, \\
&= \frac{\cos 2x}{\cos 6x} + \frac{\cos 4x}{2\cos 6x.\cos 2x} \\
\therefore \lim_{x\to 0} \frac{\tan 6x}{\sin 4x} &= \frac{1}{1} + \frac{1}{2\times 1\times 1}, \\
&= \frac{3}{2}.
\end{aligned}$$

4. (a) Show that there is a root to the equation $\sin x = x - \frac{1}{2}$ between $x = 0.5$ and $x = 1.8$. 4

Solution: Put $f(x) = x - \sin x - \frac{1}{2}$,

$$\begin{aligned}
f(0.5) &\approx -0.48, \\
f(1.8) &\approx 0.33.
\end{aligned}$$

$\therefore$ There is a root where $0.5 < x < 1.8$.

(b) Taking $x = 1.2$ as a first approximation to this solution, apply Newton's method once to find a closer approximation to the solution. Give your answer correct to two decimal places.

Solution: $f'(x) = 1 - \cos x$,

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)}, \\
&= 1.2 - \frac{1.2 - \sin 1.2 - 0.5}{1 - \cos 1.2}, \\
&\approx 1.56 \text{ (2 dec. pl.)}
\end{aligned}$$

5. (a) Write down a primitive function of $e^{f(x)}.f'(x)$. 4

Solution: $e^{f(x)}$ or $e^{f(x)} + c$.

(b) Hence evaluate

$$\int_{-1}^{1} \frac{2e^{\cos^{-1}x}}{\sqrt{1-x^2}}.dx$$

Solution:
$$\begin{aligned} \text{I} &= -2\int_{-1}^{1} \frac{-1\times e^{\cos^{-1}x}}{\sqrt{1-x^2}}.dx, \\ &= -2\left[e^{\cos^{-1}x}\right]_{-1}^{1}, \\ &= -2(1-e^{\pi}), \\ &= 2(e^{\pi}-1). \end{aligned}$$

6. The surface area of a cube is increasing at a rate of $12.6\,\text{cm}^2$ per second and initially each edge is 10 cm. 4

(a) If it maintains its cubic shape, find an expression in terms of t for the surface area of the cube after t seconds.

Solution: Let edge length be x, surface area be a, and volume be v.
$$\begin{aligned} \frac{da}{dt} &= 12.6, \\ a &= \int 12.6\,dt, \\ &= 12.6t + c. \\ \text{When } t &= 0, \quad a = 6\times 10^2 = 600, \\ \therefore a &= 12.6t + 600. \end{aligned}$$

(b) Hence or otherwise, find the rate at which the volume of the cube is increasing after 10 seconds.

Solution: First method—
$$\begin{aligned} v &= x^3, & a &= 6x^2, \\ \frac{dv}{dx} &= 3x^2, & \frac{da}{dx} &= 12x. \end{aligned}$$
$$\begin{aligned} \frac{dv}{dt} &= \frac{dv}{dx}\times\frac{dx}{da}\times\frac{da}{dt}, \\ &= 3x^2 \times \frac{1}{12x}\times 12.6, \\ &= 3.15x. \\ \text{When } t &= 10, \\ 6x^2 &= 12.6t + 600, \\ &= 726, \\ x^2 &= 121, \\ x &= 11 \text{ as } x > 0. \\ \therefore \frac{dv}{dt} &= 3.15\times 11, \\ &= 34.65. \end{aligned}$$
So the volume is increasing at $34.65\,\text{cm}^3/\text{s}$.

Solution: Second method—

$$\begin{aligned}
6x^2 &= 12.6t + 600,\\
x^2 &= 2.1t + 100,\\
x^3 &= (2.1t + 100)^{3/2},\\
v &= (2.1t + 100)^{3/2},\\
\frac{dv}{dt} &= 2.1 \times \tfrac{3}{2}(2.1t + 100)^{1/2},\\
&= 3.15\sqrt{2.1t + 100}.\\
\text{When } t &= 10,\\
\frac{dv}{dt} &= 3.15\sqrt{2.1(10) + 100},\\
&= 34.65.
\end{aligned}$$

So the volume is increasing at $34.65\,\text{cm}^3/\text{s}$.

Solution: Third method—

$$\frac{dv}{dt} = \frac{dv}{da} \times \frac{da}{dt}.$$

$$a = 6x^2, \qquad v = x^3,$$

$$x = \sqrt{\frac{a}{6}}. \qquad = \sqrt{\frac{a^3}{6^3}},$$

$$= \frac{a^{3/2}}{6\sqrt{6}}.$$

$$\frac{dv}{da} = \frac{3}{2}.\frac{a^{1/2}}{6\sqrt{6}},$$

$$= \frac{\sqrt{a}}{4\sqrt{6}}.$$

When $t = 10,\ a = 726,\ \therefore \frac{dv}{dt} = \frac{\sqrt{726}}{4\sqrt{6}} \times 12.6,$

$$= \frac{11}{4} \times 12.6,$$

$$= 34.65.$$

So the volume is increasing at $34.65\,\text{cm}^3/\text{s}$.

Section C

1\. (i) $LHS = \frac{(x^2+3)-(x^2+1)}{(x^2+1)(x^2+3)}$

$= \frac{2}{(x^2+1)(x^2+3)}$ [2]

$= RHS$ QED

(ii) $\int_{-\sqrt{3}}^{1} \frac{dx}{(x^2+1)(x^2+3)}$

$= \frac{1}{2}\int_{-\sqrt{3}}^{1}\left(\frac{1}{x^2+1} - \frac{1}{x^2+3}\right)dx$

$= \frac{1}{2}\left[\tan^{-1}x - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right]_{-\sqrt{3}}^{1}$

$= \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{\sqrt{3}}\cdot\frac{\pi}{6}\right) - \left(-\frac{\pi}{3} - \frac{1}{\sqrt{3}}\left(-\frac{\pi}{4}\right)\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{\pi}{6\sqrt{3}} + \frac{\pi}{3} - \frac{\pi}{4\sqrt{3}}\right]$

$= \frac{\pi}{2}\left(\frac{7}{12} - \frac{4+6}{24\sqrt{3}}\right)$

$= \frac{\pi}{2}\left(\frac{7}{12} - \frac{5}{12\sqrt{3}}\right)$

$= \frac{\pi}{2}\left(\frac{7\sqrt{3}-5}{12\sqrt{3}}\right)$ [3]

2\. (i) $p(n)$: $x \mid (x+1)^n - 1$ if $x \in J^+$

$p(1)$: If $n=1$, $(x+1)^n - 1 = x$ which is divisible by x.

$\therefore p(1)$ is true.

$p(k)$: Assume true for $n=k$

i.e. $x \mid (x+1)^k - 1$

$\therefore (x+1)^k - 1 = Rx$, some $R \in J$

$p(k+1)$: RTP that $p(k) \to p(k+1)$

Now when $n = k+1$

$(x+1)^{k+1} - 1$

$= (x+1)^k(x+1) - 1$

$= (Rx+1)(x+1) - 1$ by assumption

$= Rx^2 + Rx + x + 1 - 1$

$= x(Rx + R + 1)$

which is divisible by x.

$\therefore$ since $p(1)$ is true and $p(k) \to p(k+1)$ then by PMI, $p(n)$ is true. [3]

(ii) $15^n - 5^n - 3^n + 1$

$= 5^n(3^n - 1) - (3^n - 1)$

$= (5^n - 1)(3^n - 1)$

$= 4R \cdot 2S$ (say) $R, S \in J$

$= 8(RS)$ QED [2]

3. $\ddot{x} = 2x^3 - 10x$

$v = 0$, $x = -1$ when $t = 0$

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 - 10x$

$\therefore \frac{1}{2}v^2 = \int (2x^3 - 10x)\,dx + C$

$= \frac{2x^4}{4} - \frac{10x^2}{2} + C$

$v^2 = x^4 - 10x^2 + C'$

$v = 0$ when $x = -1$, so

$0 = 1 - 10 + C'$

$C' = 9$

So $v^2 = x^4 - 10x^2 + 9$

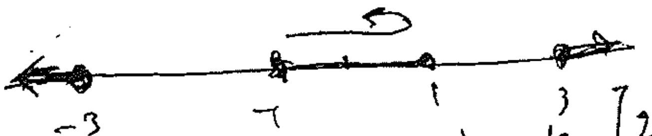
$v = \pm\sqrt{x^4 - 10x^2 + 9}$

$= \pm\sqrt{(x^2 - 9)(x^2 - 1)}$ [3]

(ii) $\therefore x^2 \leq 1$ or $x^2 \geq 9$

$\therefore (x^2 - 9)(x^2 - 1) \geq 0$

$\therefore -1 \leq x \leq 1$ or $x \leq -3$ or $x \geq 3$



ie Since particle starts at -1, it is limited to $-1 \leq x \leq 1$ [2]

(iii) If $v = 0$ when $x = 0$ then the acceleration is zero, so the particle does not move. [1]

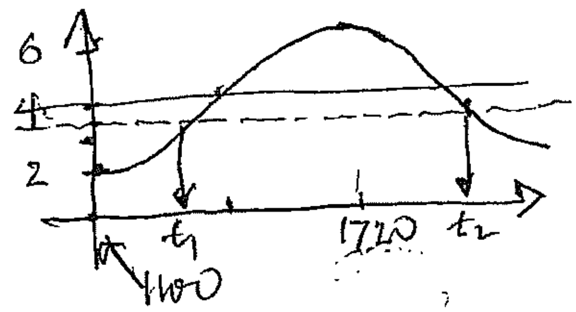
4.

1100 2m low

1720 6m High $a = 2$

+620

2340 low



Period 12:40 = 760 min. = $12\frac{2}{3}$ hrs

"Equilibrium" $x = 4$ m

$\left(760 = \frac{2\pi}{n}\right.$ $12\frac{2}{3} = \frac{2\pi}{n}$

$\left.n = \frac{2\pi}{760}\right)$ $n = \frac{2\pi}{12\frac{2}{3}}$

$(= 0.008267)$ $= 0.49604$

Hence $x = -2\cos(nt) + 4$

For $x = 3.5$ m

$3.5 = -2\cos(nt) + 4$

$-0.5 = -2\cos(nt)$

$+0.25 = \cos(nt)$

$nt = \cos^{-1}(0.25)$

$t_1 = 2h\,39' + 11{:}00$

$= 13{:}39$

$t_2 = 23{:}40 - 2{:}39$

$= 21{:}01$ [6]