SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

2013
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #3

Mathematics Extension 1

General Instructions:

- Reading time – 5 minutes
- Working time – 1.5 hours
- Write using black or blue pen
  Black pen is preferred
- Board approved calculators may be used
- A table of standard integrals is provided on the back of the Multiple Choice answer sheet
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may NOT be awarded for messy or badly arranged work
- Answer in simplest exact form unless otherwise stated

Total marks – 80 Marks
Section I Pages 2–3
8 marks

- Attempt Questions 1–8
- Answer on the Multiple Choice answer sheet provided.
- Allow about 10 minutes for this section

Section II Pages 4–7
72 marks

- Attempt Questions 9–12
- Allow about 1 hour 20 minutes for this section
- For Questions 9–12, start a new answer booklet per question

Examiner: Mr D. Hespe

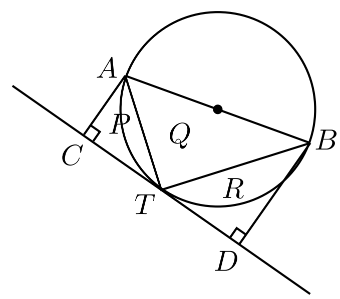
This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Section I— 8 marks
Select the alternative A, B, C, or D that best answers the question.
Fill in the response oval on your multiple choice answer sheet.

Marks

1. If $A+B=\frac{\pi}{2}$, then $\sin A\cos B+\sin B\cos A=$ $\boxed{1}$
   (A) $\sqrt{2}$
   (B) 1
   (C) 0
   (D) cannot be determined.

2. In the figure, AB is a diameter, CTD is a tangent. $AC\perp CD$; $BD\perp CD$, $\triangle ACT$, $\triangle ATB$, and $\triangle BDT$ are denoted by P, Q, and R respectively. Which of the following is a true statement about the triangles? $\boxed{1}$



   (A) No two of them are similar.
   (B) Only P and Q are similar.
   (C) Only P and R are similar.
   (D) All three of them are similar.

3. A security code consists of three letters followed by one digit. The first letter in the code must be a vowel. How many different security codes are possible? $\boxed{1}$
   (A) 33 800
   (B) 175 760
   (C) 141 960
   (D) 3390

4. $\displaystyle\int_0^{\frac{\pi}{4}}\tan^2 x\,dx=$ $\boxed{1}$
   (A) $\frac{\pi}{4}-1$
   (B) $1-\frac{\pi}{4}$
   (C) $\frac{1}{3}$
   (D) $\frac{\pi}{4}+1$

5. Newton's method for finding the square root of a real number R from the equation $x^2 - R = 0$ gives $\boxed{1}$

(A) $x_{i+1} = \dfrac{x_i}{2}$

(B) $x_{i+1} = \dfrac{3x_i}{2}$

(C) $x_{i+1} = \frac{1}{2}\left(x_i + \dfrac{R}{x_i}\right)$

(D) $x_{i+1} = \frac{1}{2}\left(3x_i - \dfrac{R}{x_i}\right)$

6. $\displaystyle\int_{-1}^{2} \frac{|x|}{x} dx =$ $\boxed{1}$

(A) 1

(B) 2

(C) 3

(D) does not exist.

7. The value of $\sin^{-1}(\sin 10)$ is $\boxed{1}$

(A) 10

(B) $10 - 3\pi$

(C) $3\pi - 10$

(D) none of these.

8. The substitution $u = \ln x$ transforms the integral $\displaystyle\int_1^e \frac{1 - \ln x}{x^2} dx$ into $\boxed{1}$

(A) $\displaystyle\int_0^1 (1 - u) du$

(B) $\displaystyle\int_0^1 (1 - u)e^{-u} du$

(C) $\displaystyle\int_1^e (1 - u)e^{u} du$

(D) $\displaystyle\int_1^e (1 - u)e^{-u} du$

Section II— 72 marks

Marks

Question 9 (18 marks) (use a separate answer booklet)

(a) Differentiate $\log_e\left(\sin^3 x\right)$ writing your answer in the simplest form. $\boxed{2}$

(b) Use the substitution $u = x^2$ to find $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}\,dx$. $\boxed{3}$

(c) Mr Dowdell has to organise a Year 11 examination timetable with seven examinations. Of these examinations, one is English and two are Mathematics. The two unit and extension Mathematics examinations are not to be scheduled consecutively. $\boxed{3}$

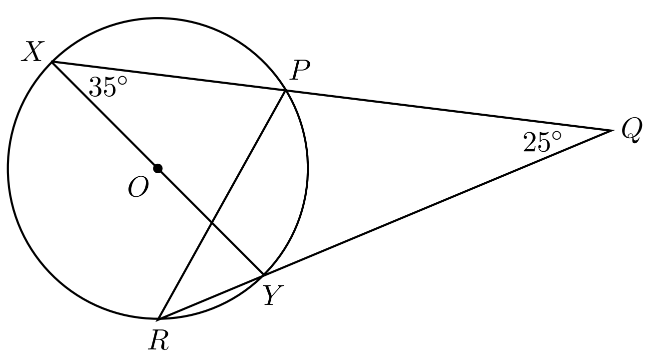
(i) In how many ways can the seven examinations be scheduled?

(ii) If the English examination is scheduled first, find the probability that one Mathematics examination will be scheduled second and the other Mathematics examination scheduled last.

(d) (i) Show that $f(x) = x - e^{-x}$ has a zero between $x = 0$ and $x = 1$. $\boxed{3}$

(ii) Use Newton's method to find a second approximation to a root of $x - e^{-x} = 0$, given that $x = 0.5$ is the first approximation.
Give the answer correct to three decimal places.

(e)



O is the centre of the circle, $\angle PXY = 35^\circ$ and $\angle PQY = 25^\circ$. $\boxed{3}$

(i) Copy the diagram onto your solution page.

(ii) Find $\angle RPY$, giving full reasons.

(f) (i) If $f(x) = e^{x+2}$, find the inverse function $f^{-1}(x)$. $\boxed{4}$

(ii) State the domain and range of $f^{-1}(x)$.

(iii) Sketch $f(x)$ and $f^{-1}(x)$ on the same axes, showing essential features.

Marks

Question 10 (18 marks) (use a separate answer booklet)

(a) A teacher is making a multiple choice test for 27 students. She wants to give each student the same questions but have each student's questions appear in a different order. What is the least number of questions the test must contain? $\boxed{2}$

(b) Evaluate $\displaystyle\int_2^3 10^{-x}\,dx$. $\boxed{2}$

(c) Find an equivalent expression to $\sec\left(\sin^{-1}(2x-1)\right)$ which does not involve trigonometric expressions. $\boxed{2}$

(d) (i) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \cos x \sin^2 x\,dx$. $\boxed{8}$

(ii) Integrate $\displaystyle\int \sec^2 x \tan x\,dx$.

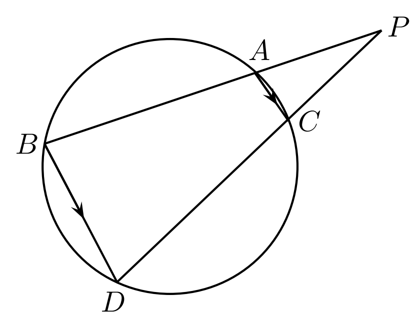
(iii) Evaluate $\displaystyle\int_0^{\frac{2\pi}{3}} \sin^2 x\,dx$.

(iv) Integrate $\displaystyle\int \sin^3 x\,dx$.

(e) $AC \parallel BD$. $\boxed{4}$

(i) Prove that $PB = PD$.

(ii) Prove $AB = CD$.



Marks

Question 11 (18 marks) (use a separate answer booklet)

(a) If ${}^{n}\mathrm{P}_r = 6720$ and ${}^{n}\mathrm{C}_r = 56$, then find the value of r. $\boxed{2}$

(b) Two concentric circles have radii r and R for the smaller and larger respectively. A chord of the outer circle is tangent to the inner circle. Find the length of this chord. $\boxed{2}$

(c) (i) Divide $x^4 + x^3$ by $x^2 + 1$. $\boxed{4}$

(ii) Hence or otherwise, integrate $\displaystyle\int \frac{x^4 + x^3}{x^2 + 1}\, dx$.

(d) (i) On the same axes, sketch $y = \ln x$ and $y = cosx$ over the range $0 < x \leqslant \pi$. $\boxed{4}$

(ii) From your sketch, make an approximation to the root of $\cos x = \ln x$ and use two iterations of Newton's method to get a better approximation correct to 4 significant figures.

(e) Find the following indefinite integrals using the substitution given. $\boxed{6}$

(i) $\displaystyle\int \frac{dx}{x(\ln x)^3}$ (Let $u = \ln x$).

(ii) $\displaystyle\int \frac{\left(\tan^{-1} x\right)^2}{1 + x^2}\, dx$ (Let $u = tan^{-1}x$).

(iii) $\displaystyle\int \frac{e^x\, dx}{\sqrt{49 - e^{2x}}}$ (Let $u = e^x$).

Marks

Question 12 (18 marks) (use a separate answer booklet)

(a) In arranging the letters of the word **AMAZED**, in how many cases is the **E** situated between the **A**s? $\boxed{2}$

(b) A submarine telegraph cable consists of a copper core with a concentric sheath of non-conducting material. The ratio of the radius of the core to the thickness of the sheath is $x : 1$, and it is known that the speed of signalling is equal to $kx^2 \ln\left(\frac{1}{x}\right)$, where k is a constant. $\boxed{3}$

Show that the greatest speed of signalling is reached in a cable for which $x = \dfrac{1}{\sqrt{e}}$.

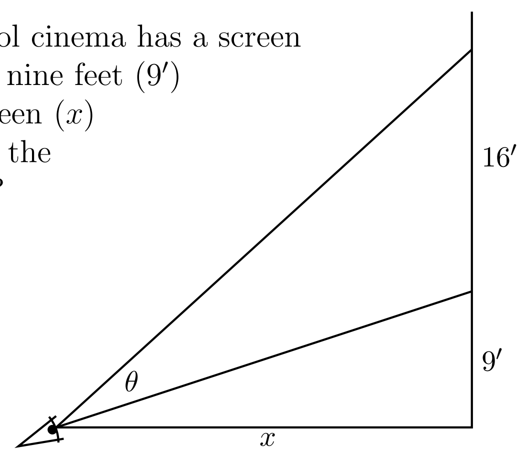
(c) (i) Find $\dfrac{d}{dx}\big[\ln(3 + 4\tan x)\big]$. $\boxed{4}$

(ii) Hence find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3 + 4\tan x}\, dx$.

(d) If $\dfrac{d^2y}{dx^2} = 2e^x - 3e^{-x}$, and when $x = 0$, $y = 4$, $\dfrac{dy}{dx} = 5$, $\boxed{4}$

find $\dfrac{dy}{dx}$ when $y = 0$.

(e) In an American high school, the school cinema has a screen sixteen feet (16′) high which is raised nine feet (9′) above eye level. How far from the screen (x) should you sit to get the best view of the screen (*i.e.* to maximise the angle θ)? $\boxed{5}$



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SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

2013
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #3

Mathematics Extension 1
Solutions

Section I— 8 marks

Select the alternative A, B, C, or D that best answers the question.
Fill in the response oval on your multiple choice answer sheet.

Marks

1. If $A+B=\frac{\pi}{2}$, then $\sin A\cos B+\sin B\cos A=$ $\boxed{1}$

   (A) $\sqrt{2}$

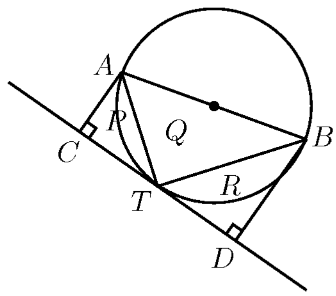
   (B) 1

   (C) 0

   (D) cannot be determined.

Solution: As the triangle is right-angled,
$\sin A=\cos B$ and $\cos A=\sin B$.
So $\sin^2 A+\cos^2 A=1$, *i.e.* B.

2. In the figure, AB is a diameter, CTD is a tangent. $AC\perp CD$; $BD\perp CD$, $\triangle ACT$, $\triangle ATB$, and $\triangle BDT$ are denoted by P, Q, and R respectively. Which of the following is a true statement about the triangles? $\boxed{1}$



   (A) No two of them are similar.

   (B) Only P and Q are similar.

   (C) Only P and R are similar.

   (D) All three of them are similar.

Solution: $\angle ATB=90°$,
$\angle ATC=\angle ATB$,
$\angle DTB=\angle TAB$,
so D.

3. A security code consists of three letters followed by one digit. The first letter in the code must be a vowel. How many different security codes are possible? $\boxed{1}$

   (A) 33 800

   (B) 175 760

   (C) 141 960

   (D) 3390

Solution: 5 vowels, 26 letters, and 10 digits to choose from,
so $5 \times 26 \times 26 \times 10 = 33\,800$, *i.e.* A.

4. $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x \, dx =$ $\boxed{1}$

(A) $\frac{\pi}{4} - 1$

(B) $1 - \frac{\pi}{4}$

(C) $\frac{1}{3}$

(D) $\frac{\pi}{4} + 1$

Solution:
$$\begin{aligned} \int_0^{\frac{\pi}{4}} \left(\sec^2 x - 1\right) dx &= \Big[\tan x - x\Big]_0^{\frac{\pi}{4}}, \\ &= 1 - \tfrac{\pi}{4} - (0 - 0), \end{aligned}$$
so B.

5. Newton's method for finding the square root of a real number R from the equation $x^2 - R = 0$ gives $\boxed{1}$

(A) $x_{i+1} = \dfrac{x_i}{2}$

(B) $x_{i+1} = \dfrac{3x_i}{2}$

(C) $x_{i+1} = \frac{1}{2}\left(x_i + \dfrac{R}{x_i}\right)$

(D) $x_{i+1} = \frac{1}{2}\left(3x_i - \dfrac{R}{x_i}\right)$

Solution:
$$\begin{aligned} \text{We know } x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)} \\ \text{where } f(x) &= x^2 - R, \quad f'(x) = 2x. \\ \text{Thus } x_{i+1} &= x_i - \frac{x_i^2 - R}{2x_i}, \\ &= x_i - \frac{x_i}{2} + \frac{R}{2x_i}, \\ &= \tfrac{1}{2}\left(x_i + \frac{R}{x_i}\right), \text{ so C.} \end{aligned}$$

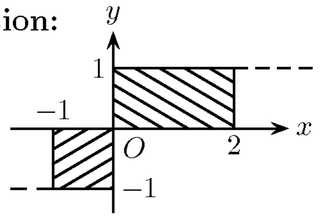
6. $\displaystyle\int_{-1}^{2} \frac{|x|}{x} dx =$ $\boxed{1}$

(A) 1

(B) 2

(C) 3

(D) does not exist.

Solution: 

Using a simple graph, it is clear that $2-1=1$, so A.

7. The value of $\sin^{-1}(\sin 10)$ is $\boxed{1}$

(A) 10

(B) $10-3\pi$

(C) $3\pi-10$

(D) none of these.

Solution: Note that 10 is a little bigger than 3π so is in the third quadrant. The domain of $\sin^{-1}x$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, and the equivalent angle is in the fourth quadrant, $3\pi-10$, thus C (or just experiment with the calculator).

8. The substitution $u=\ln x$ transforms the integral $\displaystyle\int_1^e \frac{1-\ln x}{x^2}dx$ into $\boxed{1}$

(A) $\displaystyle\int_0^1 (1-u)du$

(B) $\displaystyle\int_0^1 (1-u)e^{-u}du$

(C) $\displaystyle\int_1^e (1-u)e^{u}du$

(D) $\displaystyle\int_1^e (1-u)e^{-u}du$

Solution:
$$\begin{aligned}\frac{du}{dx} &= \frac{1}{x},\\ dx &= x\,du,\\ &= e^u du.\\ \text{When } x &= 1,\quad u=0,\\ x &= e,\quad u=1.\end{aligned}$$

So $\displaystyle\int_0^1 \frac{1-u}{e^{2u}}e^u du = \int_0^1 (1-u)e^{-u}du$, *i.e.* B.

Section II— 72 marks

Marks

Question 9 (18 marks) (use a separate answer booklet)

(a) Differentiate $\log_e\left(\sin^3 x\right)$ writing your answer in the simplest form. $\boxed{2}$

Solution:
$$\begin{aligned}\frac{d}{dx}\left(\log_e\left(\sin^3 x\right)\right) &= \frac{\frac{d}{dx}\left(\sin^3 x\right)}{\sin^3 x},\\ &= \frac{3\sin^2 x.\cos x}{sin^3 x},\\ &= \frac{3\cos x}{\sin x},\\ &= 3\cot x.\end{aligned}$$

(b) Use the substitution $u = x^2$ to find $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}\,dx$. $\boxed{3}$

Solution:
$$\begin{aligned}\text{Let } u &= x^2, & \therefore du &= 2x.dx.\\ \text{When } x &= 0, & u &= 0,\\ x &= \tfrac{1}{\sqrt{2}}, & u &= \tfrac{1}{2}.\end{aligned}$$
$$\begin{aligned}\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}\,dx &= \tfrac{1}{2}\int_0^{\frac{1}{2}} \frac{2x.dx}{\sqrt{1-x^4}},\\ &= \tfrac{1}{2}\int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}},\\ &= \tfrac{1}{2}\Big[\sin^{-1} u\Big]_0^{\frac{1}{2}},\\ &= \tfrac{1}{2}\left(\frac{\pi}{6}-0\right),\\ &= \frac{\pi}{12}.\end{aligned}$$

(c) Mr Dowdell has to organise a Year 11 examination timetable with seven examinations. Of these examinations, one is English and two are Mathematics. The two unit and extension Mathematics examinations are not to be scheduled consecutively. $\boxed{3}$

(i) In how many ways can the seven examinations be scheduled?

Solution:
$$\begin{aligned}\text{Total arrangements} &= 7!\\ \text{Ways with Maths together} &= 2\times 6!\\ \text{Ways with Maths separated} &= 7! - 2\times 6!\\ &= 3600.\end{aligned}$$
I.e. there are 3600 ways of scheduling the exams.

(ii) If the English examination is scheduled first, find the probability that one Mathematics examination will be scheduled second and the other Mathematics examination scheduled last.

Solution: One interpretation—

$$\begin{aligned}
\text{Ways of arranging Maths} &= 2. \\
\text{The remaining 4 subjects} &= 4! \\
\text{So total ways} &= 48. \\
\text{Ways of arranging 6 exams} &= 6!, \\
\text{so the probability} &= \frac{48}{720}, \\
&= \frac{1}{15}.
\end{aligned}$$

Another interpretation—

$$\begin{aligned}
\text{Total ways} &= 48 \text{ as above.} \\
\text{Ways of arranging 7 exams} &= 3600 \text{ from part (i)}, \\
\text{so the probability} &= \frac{48}{3600}, \\
&= \frac{1}{75}.
\end{aligned}$$

(d) (i) Show that $f(x) = x - e^{-x}$ has a zero between $x = 0$ and $x = 1$. [3]

Solution:

$$\begin{aligned}
f(0) &= 0 - 1, \\
&= -1 < 0, \\
f(1) &\approx 1 - 0.368, \\
&\approx 0.632 > 0.
\end{aligned}$$

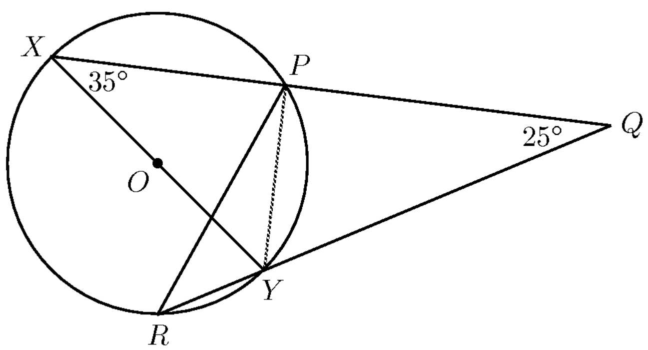
As the function is continuous, there must be a zero on the interval $(0, 1)$.

(ii) Use Newton's method to find a second approximation to a root of $x - e^{-x} = 0$, given that $x = 0.5$ is the first approximation.
Give the answer correct to three decimal places.

Solution:

$$\begin{aligned}
\text{As } f(x) &= x - e^{-x}, \\
f'(x) &= 1 + e^{-x}. \\
\text{Now } x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)}, \\
&= 0.5 - \frac{0.5 - 0.6065}{1 + 0.6065}, \\
&= 0.5663110032 \text{ (by calculator)}.
\end{aligned}$$

So the second approximation is 0.566 (3 dec. pl.).

(e) 

O is the centre of the circle, $\angle PXY = 35°$ and $\angle PQY = 25°$. [3]

(i) Copy the diagram onto your solution page.

(ii) Find $\angle RPY$, giving full reasons.

Solution:
$$\begin{aligned} \angle PRY &= 35° \text{ (angle in the same segment)},\\ \angle XPR &= \angle PRQ + \angle PQR \text{ (exterior angle of } \triangle PRQ),\\ &= 35° + 25°,\\ &= 60°.\\ \text{Now } \angle XPY &= 90° \text{ (angle in semicircle)},\\ \angle RPY &= 90° - \angle XPR.\\ &= 30°. \end{aligned}$$

(f) (i) If $f(x) = e^{x+2}$, find the inverse function $f^{-1}(x)$. $\boxed{4}$

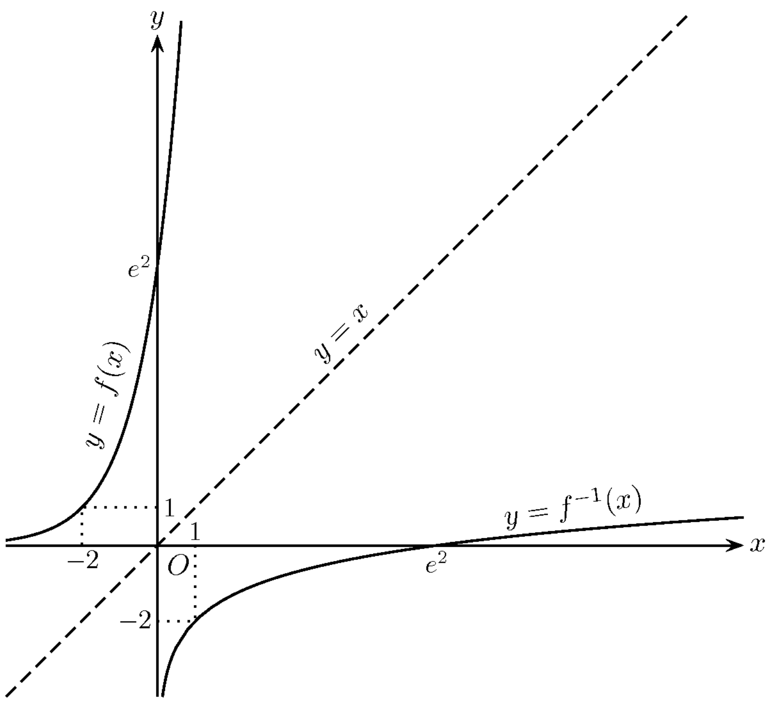
Solution:
$$\begin{aligned} \text{Put } y &= e^{x+2},\\ \text{then for the inverse, } x &= e^{y+2},\\ \ln x &= y + 2,\\ y &= \ln x - 2,\\ \text{so the inverse is } f^{-1}(x) &= \ln x - 2. \end{aligned}$$

(ii) State the domain and range of $f^{-1}(x)$.

Solution: Domain: $x > 0$,
Range: $f^{-1}(x) \in \mathbb{R}$.

(iii) Sketch $f(x)$ and $f^{-1}(x)$ on the same axes, showing essential features.

Solution:



Marks

Question 10 (18 marks) (use a separate answer booklet)

(a) A teacher is making a multiple choice test for 27 students. She wants to give each student the same questions but have each student's questions appear in a different order. What is the least number of questions the test must contain? $\boxed{2}$

Solution: If 2 questions, $2! = 2$ arrangements, (too few).
If 3 questions, $3! = 6$ arrangements, (too few).
If 4 questions, $4! = 24$ arrangements, (too few).
If 5 questions, $5! = 120$ arrangements, which is more than enough.

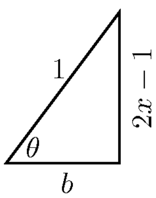
(b) Evaluate $\displaystyle\int_2^3 10^{-x}\,dx$. $\boxed{2}$

Solution:

$$\begin{aligned}\int_2^3 e^{-x\ln 10}\,dx &= \left[\frac{e^{-x\ln 10}}{-\ln 10}\right]_2^3, \qquad \left(10^{-x} = e^{-x\ln 10}\right)\\ &= \frac{e^{-3\ln 10} - e^{-2\ln 10}}{-\ln 10},\\ &= \frac{10^{-2} - 10^{-3}}{\ln 10},\\ &= \frac{0.009}{\ln 10}.\end{aligned}$$

(c) Find an equivalent expression to $\sec\left(\sin^{-1}(2x-1)\right)$ which does not involve trigonometric expressions. $\boxed{2}$

Solution:



$$\begin{aligned}b^2 &= 1^2 - (2x-1)^2,\\ &= 1 - 4x^2 + 4x - 1,\\ &= 4x - 4x^2,\\ b &= 2\sqrt{x - x^2} \text{ (taking positive as the domain of secant is positive over the range of } \sin^{-1}).\\ \therefore \sec\theta &= \frac{1}{2\sqrt{x-x^2}}.\end{aligned}$$

(d) (i) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \cos x \sin^2 x\,dx$. $\boxed{8}$

Solution:

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \cos x\sin^2 x\,dx &= \int_0^{\frac{\pi}{4}} \sin^2 x\,d\sin x,\\ &= \left[\frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{4}},\\ &= \frac{1}{6\sqrt{2}},\\ &= \frac{\sqrt{2}}{12}.\end{aligned}$$

(ii) Integrate $\displaystyle\int \sec^2 x \tan x \, dx$.

Solution: $$\begin{aligned}\int \sec^2 x \tan x \, dx &= \int \sec x \, d\sec x, \\ &= \frac{\sec^2 x}{2} + c.\end{aligned}$$

(iii) Evaluate $\displaystyle\int_0^{\frac{2\pi}{3}} \sin^2 x \, dx$.

Solution: $\cos 2x = 1 - 2\sin^2 x$, so $\sin^2 x = \dfrac{1-\cos 2x}{2}$.
$$\begin{aligned}\int_0^{\frac{2\pi}{3}} \frac{1-\cos 2x}{2} \, dx &= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{2\pi}{3}}, \\ &= \frac{1}{2}\left(\frac{2\pi}{3} - \frac{-\sqrt{3}}{2\times 2} - (0-0)\right), \\ &= \frac{8\pi + 3\sqrt{3}}{24}.\end{aligned}$$

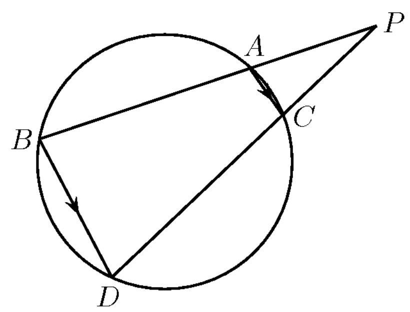
(iv) Integrate $\displaystyle\int \sin^3 x \, dx$.

Solution: $$\begin{aligned}\int \sin^2 x \sin x \, dx &= \int (1-\cos^2 x) \, d(-\cos x), \\ &= \frac{\cos^3 x}{3} - \cos x + c.\end{aligned}$$

(e) $AC \parallel BD$. $\boxed{4}$

(i) Prove that $PB = PD$.

(ii) Prove $AB = CD$.



Solution: (i) $\angle PAC = \angle PBD$ (corresp. $\angle$s, $AC \parallel BD$),
$\angle PAC = \angle CDB$ (exterior angle equals interior opposite angle of cyclic quadrilateral $ACDB$),
$\therefore \triangle PBD$ is isosceles (base angles: $\angle PBD = \angle CDB$),
so $PB = PD$ (sides opposite equal $\angle$s, isosc. $\triangle PBD$).

$$\begin{aligned}
\text{(ii)}\quad PA \times PB =&\ PC \times PD \text{ (intersecting secants theorem),}\\
\text{but } PB =&\ PD \text{ (shown above),}\\
\text{thus } PA =&\ PC,\\
\text{and } AB =&\ PB - PA, \quad CD = PD - PC,\\
\text{so } AB =&\ CD.
\end{aligned}$$

Marks

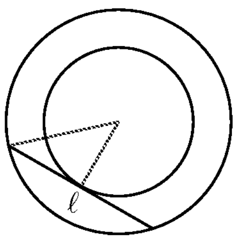
Question 11 (18 marks) (use a separate answer booklet)

(a) If ${}^{n}\mathrm{P}_r = 6720$ and ${}^{n}\mathrm{C}_r = 56$, then find the value of r. $\boxed{2}$

Solution:
$$\begin{aligned} \frac{n!}{(n-r)!} &= 6720, \\ \frac{n!}{r!(n-r)!} &= 56, \\ \therefore 56 &= \frac{6720}{r!}, \\ r! &= \frac{6720}{56}, \\ &= 120 = 5! \\ \text{So } r &= 5. \end{aligned}$$

(b) Two concentric circles have radii r and R for the smaller and larger respectively. A chord of the outer circle is tangent to the inner circle. Find the length of this chord. $\boxed{2}$

Solution:



$$\begin{aligned} \left(\frac{\ell}{2}\right)^2 &= R^2 - r^2, \\ \ell^2 &= 4\big(R^2 - r^2\big), \\ \ell &= 2\sqrt{R^2 - r^2}. \end{aligned}$$

(c) (i) Divide $x^4 + x^3$ by $x^2 + 1$. $\boxed{4}$

Solution:
$$\begin{array}{r|rrrrr} & x^2 & +x & -1 & & \\ \hline x^2+1 & x^4 & +x^3 & +0x^2 & +0x & +0 \\ & x^4 & & +x^2 & & \\ \hline & & x^3 & -x^2 & +0x & \\ & & x^3 & & +x & \\ \hline & & & -x^2 & -x & +0 \\ & & & -x^2 & & -1 \\ \hline & & & & -x & +1 \end{array}$$

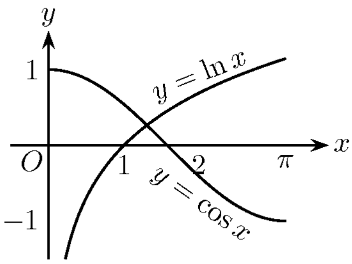
$$\therefore \frac{x^4 + x^3}{x^2 + 1} = x^2 + x - 1 - \frac{x-1}{x^2+1}.$$

(ii) Hence or otherwise, integrate $\displaystyle\int \frac{x^4 + x^3}{x^2 + 1}\,dx$.

Solution:
$$\begin{aligned} \mathrm{I} &= \int \left(x^2 + x - 1 - \frac{x-1}{x^2+1} \right) dx, \\ &= \int \left(x^2 + x - 1 - \frac{1}{2} \times \frac{2x}{x^2+1} + \frac{1}{x^2+1} \right) dx, \\ &= \frac{x^3}{3} + \frac{x^2}{2} - x - \tfrac{1}{2}\ln(x^2+1) + \tan^{-1} x + c. \end{aligned}$$

(d) (i) On the same axes, sketch $y = \ln x$ and $y = cosx$ over the range $0 < x \leqslant \pi$. 4

Solution:



(ii) From your sketch, make an approximation to the root of $\cos x = \ln x$ and use two iterations of Newton's method to get a better approximation correct to 4 significant figures.

Solution:
$$\begin{aligned} \text{Put } y &= \cos x - \ln x, \\ \text{then } y' &= -sinx - \tfrac{1}{x}. \\ x_0 &\approx 1.25 \text{ from my sketch}, \\ x_1 &= 1.25 + \frac{\cos 1.25 - \ln 1.25}{\sin 1.25 + \frac{1}{1.25}}, \\ &= 1.302704186 \text{ by calculator}, \\ x_2 &= 1.3027 + \frac{\cos 1.3027 - \ln 1.3027}{\sin 1.3027 + \frac{1}{1.3027}}, \\ &= 1.302963995 \text{ by calculator}, \\ &\approx 1.303 \text{ to 4 sig. fig.} \end{aligned}$$

(e) Find the following indefinite integrals using the substitution given. 6

(i) $\displaystyle\int \frac{dx}{x(\ln x)^3}$ (Let $u = \ln x$).

Solution:
$$\begin{aligned} \frac{du}{dx} &= \frac{1}{x}, \ i.e.\ du = \frac{1}{x}\,dx. \\ \int \frac{1}{(\ln x)^3}.\frac{1}{x}\,dx &= \int \frac{1}{u^3}\,du, \\ &= \int u^{-3}\,du, \\ &= \frac{u^{-2}}{-2} + c, \\ &= -\frac{1}{2(\ln x)^2} + c. \end{aligned}$$

(ii) $\displaystyle\int \frac{\left(\tan^{-1} x\right)^2}{1+x^2}\,dx$ (Let $u = tan^{-1}x$).

Solution: $\displaystyle \frac{du}{dx} = \frac{1}{1+x^2},\ i.e.\ du = \frac{1}{1+x^2}\,dx.$

$$\begin{aligned}\int \left(\tan^{-1} x\right)^2 . \frac{1}{1+x^2}\,dx &= \int u^2\,du,\\ &= \frac{u^3}{3} + c,\\ &= \tfrac{1}{3}\left(\tan^{-1} x\right)^3 + c.\end{aligned}$$

(iii) $\displaystyle\int \frac{e^x\,dx}{\sqrt{49-e^{2x}}}$ (Let $u = e^x$).

Solution: $\displaystyle \frac{du}{dx} = e^x,\ i.e.\ du = e^x\,dx.$

$$\begin{aligned}\int \frac{1}{\sqrt{49-e^{2x}}}.e^x\,dx &= \int \frac{1}{\sqrt{49-u^2}}\,du,\\ &= \sin^{-1}\left(\frac{u}{7}\right) + c,\\ &= \sin^{-1}\left(\tfrac{1}{7}e^x\right) + c.\end{aligned}$$

Marks

Question 12 (18 marks) (use a separate answer booklet)

(a) In arranging the letters of the word **AMAZED**, in how many cases is the **E** situated between the **A**s? $\boxed{2}$

> **Solution:** **AAE**, **AEA**, and **EAA** are the only possible arrangements of these three letters, so one third of the total will have the desired arrangement.
> $\frac{1}{3} \times \frac{6!}{2!} = 120.$
> There are 120 cases that satisfy the given condition.

(b) A submarine telegraph cable consists of a copper core with a concentric sheath of non-conducting material. The ratio of the radius of the core to the thickness of the sheath is $x : 1$, and it is known that the speed of signalling is equal to $kx^2 \ln\left(\frac{1}{x}\right)$, where k is a constant. $\boxed{3}$

Show that the greatest speed of signalling is reached in a cable for which $x = \dfrac{1}{\sqrt{e}}$.

> **Solution:** Put speed $= v$,
> $$\begin{aligned} v &= kx^2 \ln \tfrac{1}{x}, \\ \frac{dv}{dx} &= 2kx \ln \tfrac{1}{x} + kx^2 \times \frac{-1}{x^2} \times \frac{1}{\frac{1}{x}}, \\ &= kx\left(2\ln \tfrac{1}{x} - 1\right), \\ &= 0 \text{ when } -\ln x = \tfrac{1}{2}, \\ &\qquad i.e.\ x = e^{-\frac{1}{2}}. \\ \frac{d^2v}{dx^2} &= k\left(2\ln \tfrac{1}{x} - 1\right) + kx \times 2 \times \frac{-1}{x^2} \times \frac{1}{\frac{1}{x}}, \\ &= 2k\ln \tfrac{1}{x} - k - 2k, \\ &= k\left(2\ln \tfrac{1}{x} - 3\right), \\ &= -2k \text{ when } x = e^{-\frac{1}{2}}. \end{aligned}$$
> $\therefore$ The maximum speed occurs when $x = \dfrac{1}{\sqrt{e}}$.

(c) (i) Find $\dfrac{d}{dx}\big[\ln(3 + 4\tan x)\big]$. $\boxed{4}$

> **Solution:** $$\begin{aligned} \frac{d}{dx}\big[\ln(3+4\tan x)\big] &= 4\sec^2 x \times \frac{1}{(3+4\tan x)}, \\ &= \frac{4\sec^2 x}{(3+4\tan x)}. \end{aligned}$$

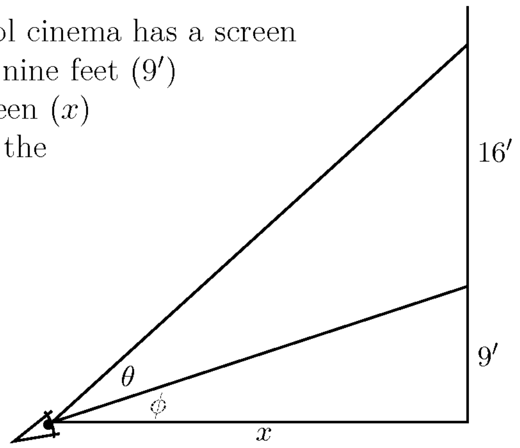
(ii) Hence find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3 + 4\tan x}\, dx$.

$$\textbf{Solution:}\quad \tfrac{1}{4}\int_0^{\frac{\pi}{4}} \frac{4\sec^2 x}{3+4\tan x}\,dx = \tfrac{1}{4}\Big[\ln(3+4\tan x)\Big]_0^{\frac{\pi}{4}},$$
$$= \tfrac{1}{4}\big(\ln(3+4)-\ln(3+0)\big),$$
$$= \tfrac{1}{4}\ln\left(\tfrac{7}{3}\right).$$

(d) If $\dfrac{d^2y}{dx^2} = 2e^x - 3e^{-x}$, and when $x=0$, $y=4$, $\dfrac{dy}{dx}=5$, find $\dfrac{dy}{dx}$ when $y=0$. 4

Solution:
$$\begin{aligned}
\frac{dy}{dx} &= 2e^x + 3e^{-x} + c,\\
5 &= 2+3+c,\\
\therefore \frac{dy}{dx} &= 2e^x + 3e^{-x},\\
y &= 2e^x - 3e^{-x} + c,\\
4 &= 2-3+c,\\
\therefore y &= 2e^x - 3e^{-x} + 5.\\
\text{Now, putting } y &= 0 \text{ and } u = e^x,\\
0 &= 2u - \frac{3}{u} + 5,\\
2u^2+5u-3 &= 0,\\
(2u-1)(u+3) &= 0,\\
u &= \tfrac{1}{2} \text{ or } -3.\\
\text{But } e^x &> 0, \text{ so } e^x = \tfrac{1}{2},\\
\text{thus } \frac{dy}{dx} &= 2\times\tfrac{1}{2} + 3\times 2 \text{ when } y=0,\\
&= 7.
\end{aligned}$$

(e) In an American high school, the school cinema has a screen sixteen feet ($16'$) high which is raised nine feet ($9'$) above eye level. How far from the screen (x) should you sit to get the best view of the screen (*i.e.* to maximise the angle θ)? 5



Solution:

$$\begin{aligned}
\tan\phi &= \frac{9}{x},\\
\tan(\theta+\phi) &= \frac{25}{x},\\
\theta &= (\theta+\phi)-\phi,\\
&= \tan^{-1}\left(\frac{25}{x}\right)-\tan^{-1}\left(\frac{9}{x}\right),\\
\text{put } y &= \tan^{-1}u, \qquad u = \frac{9}{x},\\
\frac{dy}{du} &= \frac{1}{1+u^2}. \qquad \frac{du}{dx} = \frac{-9}{x^2}.\\
\therefore \frac{dy}{dx} &= \frac{1}{1+\frac{81}{x^2}}\times\frac{-9}{x^2},\\
&= \frac{-9}{x^2+81}.\\
\frac{d\theta}{dx} &= \frac{-25}{x^2+625}-\frac{-9}{x^2+81},\\
&= \frac{-25x^2-2025+9x^2+5625}{(x^2+625)(x^2+81)},\\
&= \frac{3600-16x^2}{(x^2+625)(x^2+81)},\\
&= 0 \text{ when } 16x^2 = 3600,\\
x^2 &= 225,\\
x &= \pm 15.
\end{aligned}$$

Now, testing for a maximum,

x	14	16
$\frac{d\theta}{dx}$	2.0×10^{-3}	-1.6×10^{-3}

so the best viewing will be fifteen feet from the screen.

End of Paper