SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

HSC Assessment Task 3

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes.
- Working time – 90 Minutes.
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each question in Section II is to be answered in a separate booklet.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Answer in simplest **EXACT** form unless otherwise instructed.
- A reference sheet is provided.

Total Marks – 61

Section I (7 Marks)

Answer questions 1-7 on the Multiple Choice answer sheet provided.

Section II (54 Marks)

For Questions 8-10, start a new answer booklet for each question.

Examiner: J. Chan

Section I

10 marks
Attempt Questions 1-7

Use the multiple-choice answer sheet for Questions 1-7

1) What is the domain of the function $f(x) = 5\sin^{-1}\left(\dfrac{x}{3}\right)$?

(A) $-\dfrac{5\pi}{2} \le x \le \dfrac{5\pi}{2}$

(B) $-3 \le x \le 3$

(C) $-5 \le x \le 5$

(D) $-\dfrac{\pi}{3} \le x \le \dfrac{\pi}{3}$

2) Which of the following is the exact value of $\displaystyle\int_{\frac{3}{\sqrt{2}}}^{3} \frac{4}{\sqrt{9-x^2}}\,dx$?

(A) $-\pi$

(B) $-\dfrac{\pi}{4}$

(C) $\dfrac{\pi}{4}$

(D) π

3) Given $f(x) = \dfrac{3}{x} - 4$, $f^{-1}(4) = ?$

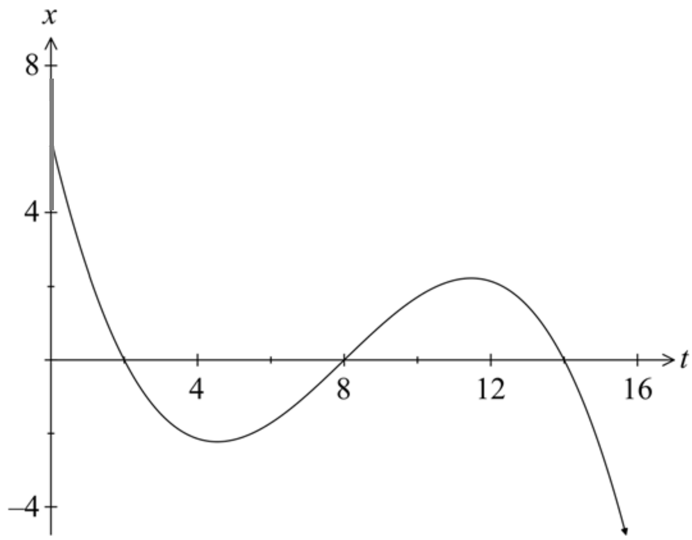
(A) $-\dfrac{13}{4}$

(B) $\dfrac{13}{4}$

(C) $\dfrac{3}{8}$

(D) $-\dfrac{3}{8}$

4) The displacement, x metres, from the origin of a particle moving in a straight line at any time (t seconds) is shown in the graph.



When was the particle at rest?

(A) $t = 0$

(B) $t = 2$, $t = 8$ and $t = 14$

(C) $t = 5$ and $t = 11$

(D) $t = 8$

5) Using the substitution $u = \log_e x$, which of the following is equal to $\int_e^{e^2} \frac{1}{x \log_e x}\, dx$?

(A) $\int_e^{e^2} \frac{du}{u}$

(B) $\int_e^{e^2} \frac{du}{e^u u}$

(C) $\int_1^2 \frac{du}{u}$

(D) $\int_1^2 \frac{du}{e^u u}$

6) The acceleration of a particle is given by $a = 6x^2 - 4x - 3$, where x is the displacement in cm. The particle initially is at the origin and has a velocity of 3 cm/s. What is the speed when the particle is at $x = 3$?

(A) $2\sqrt{7}$ cm/s

(B) $3\sqrt{7}$ cm/s

(C) $\sqrt{41}$ cm/s

(D) $\sqrt{57}$ cm/s

7) What is the value of $\cos^{-1}\left(\cos(3\pi+\alpha)\right)$ where α is an acute angle?

(A) $3\pi+\alpha$

(B) α

(C) $\pi+\alpha$

(D) $\pi-\alpha$

End of Section I

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Section II

54 marks
Attempt Questions 8–10

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

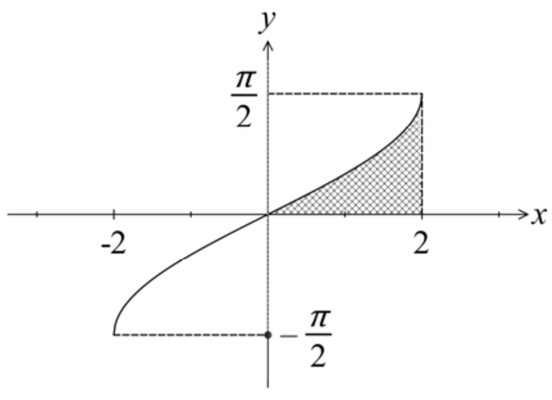
In Questions 8-10, your responses should include relevant mathematical reasoning and/or calculations.

Question 8 (16 marks) Use a SEPARATE writing booklet.

(a) Find the primitive of $2\sin^2 x - x^2$ **2**

(b) Show that $\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) = \cos^{-1}x$ **2**

(c) The shaded area is represented by $\int_0^2 \sin^{-1}\frac{x}{2}\,dx$. **2**



Explain why the shaded area is equal to $\pi - 2\int_0^{\frac{\pi}{2}} \sin y\,dy$.

Question 8 continues next page

Question 8 (continued)

(d) Consider the function $f(x) = 1 + \dfrac{2}{x-3}$ for $x > 3$.

i) Find the inverse function $f^{-1}(x)$. **1**

ii) State the domain and range of the inverse function. **2**

iii) Hence sketch $y = f^{-1}(x)$. **1**

(e) The two equal sides of an isosceles triangle are of length 6 cm. If the angle between them is increasing at the rate of 0.05 radians per second, find the rate at which the area of the triangle is increasing when the angle between the equal sides is $\dfrac{\pi}{6}$ radians. **3**

(f) A particle moves in a straight line so that at any time t, its displacement from a fixed point is x and its velocity is v. **3**

If the acceleration is $3x^2$ and $v = -\sqrt{2}$, $x = 1$ when $t = 0$, find x as a function of t.

End of Question 8

Question 9 (19 marks) Use a SEPARATE writing booklet.

(a) i) Show that $\dfrac{u^3}{u+1} = u^2 - u + 1 - \dfrac{1}{u+1}$ **1**

ii) Hence, by using the substitution $u = \sqrt{x}$, show that **3**

$$\int_0^4 \frac{x}{1+\sqrt{x}}\,dx = \frac{16}{3} - \ln 9$$

(b) i) Use the derivative of $\cos\theta$ to show that **1**

$$\frac{d}{d\theta}(\sec\theta) = \sec\theta\tan\theta$$

ii) Use the substitution $x = \sec\theta - 1$ to find the exact value of **3**

$$\int_{\sqrt{2}-1}^{1} \frac{dx}{(x+1)\sqrt{x^2+2x}}$$

(c) A particle, moving along the x-axis, starts at the origin with an initial velocity v_0.

Its acceleration is given by $\dfrac{d^2x}{dt^2} = 4x^3 - 16x$.

i) Show that the quantity $E = \dfrac{1}{2}\left(\dfrac{dx}{dt}\right)^2 - x^4 + 8x^2$ does not change with time. **1**

ii) Given that initial velocity $v_0 = \sqrt{\dfrac{31}{8}}$, find the value of E. **1**

iii) Hence, or otherwise, determine the range of the particle with the initial velocity in part (ii). **2**

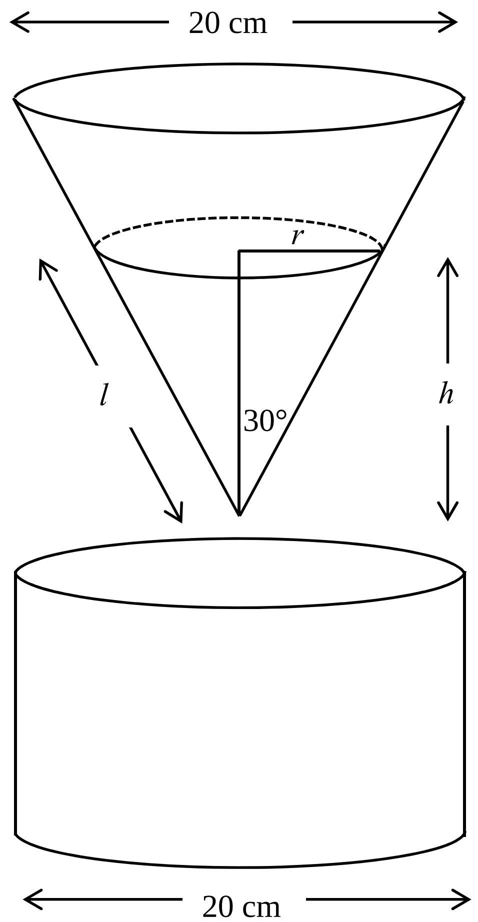
Question 9 continues next page

Question 9 (continued)

(d) An experiment is conducted using the conical filter which is held with its axis vertical as shown. The filter has a radius of 10 cm and semi-vertical angle 30°.

Chemical solution flows from the filter into the cylindrical container, with radius 10 cm, at a constant rate of 3 cm^3/s.

At time t seconds, the amount of solution in the filter has height h cm and radius r cm.



The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$ and the curved surface area is πrl where l is the slant height of the cone.

i) Find the rate of decrease (to 3 sig. fig.) of the radius of the solution in the filter when $h = 5$ cm. **3**

ii) Let S denote the curved surface area of the filter in contact with the solution. **2**

Show that $\frac{dS}{dt} = -\frac{4\sqrt{3}}{r}$ cm^2/s.

iii) When the height of the solution in the cylindrical container measures 0.81cm, the volume of the solution left in the filter and the container are the same. **2**

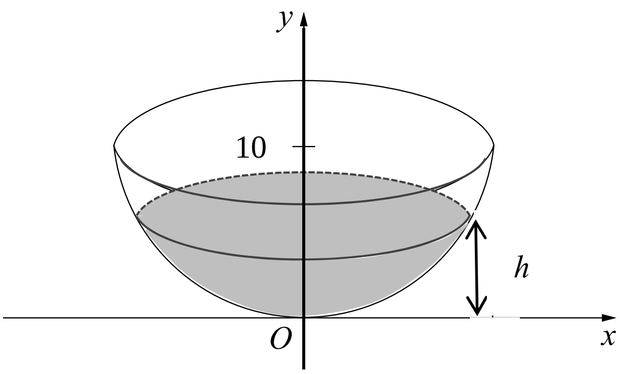
Find the rate of change with the curved surface area of the filter in contact with the solution with respect to time at this instant.

End of Question 9

Question 10 (19 marks) Use a SEPARATE writing booklet.

(a) Find the exact value of $\sin\left[\tan^{-1}\left(\frac{3}{2}\right)+\cos^{-1}\left(\frac{2}{3}\right)\right]$ **3**

(b) A hemispherical bowl of radius 10 cm is being filled with water at a constant rate of 20 cm^3 per minute.



NOT TO SCALE

i) Show that the volume of the water in the bowl in terms of its depth h is **3**

$$V=\pi\left(10h^2-\frac{h^3}{3}\right).$$

ii) At what rate (to 2 dec.pl.) is the depth of the water rising when it is 5 cm high? **3**

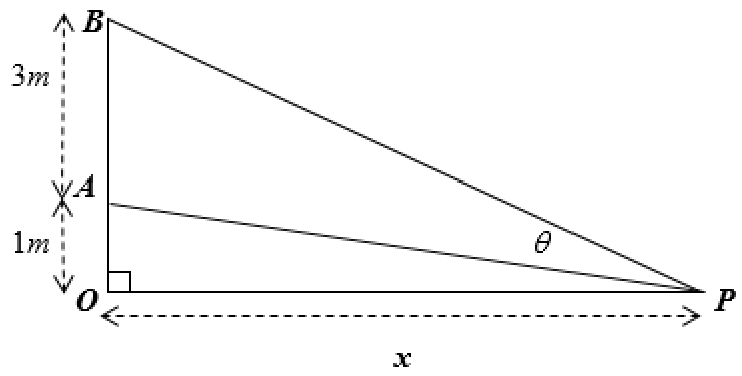
c) Let P_n denote the proposition $\sum_{r=1}^{n} r\ln\left(\frac{r+1}{r}\right)=\ln\frac{(n+1)^n}{n!}$. **4**

Prove by mathematical induction that this proposition is true for all positive integers, n.

Question 10 continues next page

Question 10 (continued)

(d) In the diagram, a vertical pole A, 3 metres high is placed on top of a support 1 metre high. The pole subtends an angle of θ radians at the point P, which is x metres from the base O of the support.



i) Show that $\theta = \tan^{-1}\dfrac{4}{x} - \tan^{-1}\dfrac{1}{x}$. **1**

ii) Show that θ is maximum when $x = 2$. **3**

iii) Deduce that the maximum angle subtended at P is $\theta = \tan^{-1}\dfrac{3}{4}$. **2**

End of paper

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SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2016

Year 12

Assessment Task 3

Mathematics Extension 1

Suggested Solutions & Markers' Comments

QUESTION	Marker
1 - 7	-
8	RB
9	BK
10	AMG

Multiple Choice Answers

1. B
2. D
3. C
4. C
5. C
6. B
7. D

1) What is the domain of the function $f(x) = 5\sin^{-1}\left(\frac{x}{3}\right)$?

$-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$.

(A) $-\frac{5\pi}{2} \le x \le \frac{5\pi}{2}$

(B) $-3 \le x \le 3$

(C) $-5 \le x \le 5$

(D) $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$

2) Which of the following is the exact value of $\int_{\frac{3}{\sqrt{2}}}^{3} \frac{4}{\sqrt{9-x^2}}\,dx$?

$4.\sin^{-1}\frac{x}{3}\Big]_{\frac{3}{\sqrt{2}}}^{3}$

$4\left[\sin^{-1}1 - \sin^{-1}\frac{1}{\sqrt{2}}\right]$

$4\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = 4.\left[\frac{2\pi-\pi}{4}\right]$

$= \pi$

(A) $-\pi$

(B) $-\frac{\pi}{4}$

(C) $\frac{\pi}{4}$

(D) π

3) Given $f(x) = \frac{3}{x} - 4$, $f^{-1}(4) = ?$

let $y = \frac{3}{x} - 4$

So $x = \frac{3}{y} - 4$

$\frac{3}{y} = \frac{x+4}{1}$

$\frac{y}{3} = \frac{1}{x+4}$

$y = \frac{3}{x+4}$

let $y = f^{-1}(4) = \frac{3}{4+4} = \frac{3}{8}$

(A) $-\frac{13}{4}$

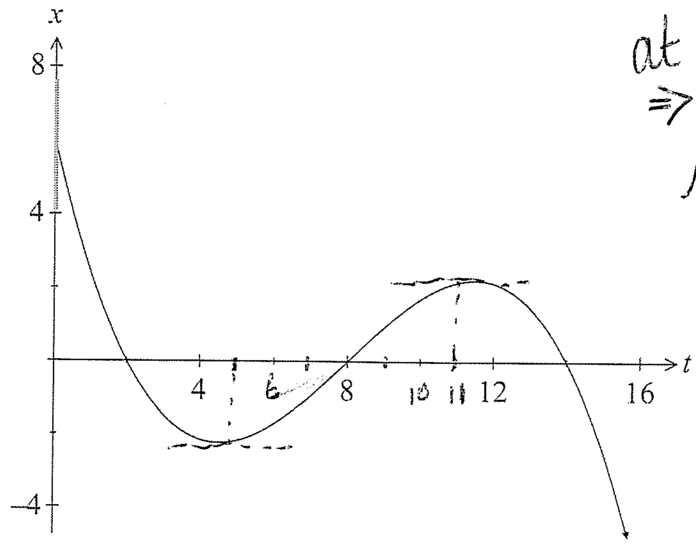
(B) $\frac{13}{4}$

(C) $\frac{3}{8}$

(D) $-\frac{3}{8}$

4) The displacement, x metres, from the origin of a particle moving in a straight line at any time (t seconds) is shown in the graph.

at rest $\Rightarrow$ slope of tangent line is horizontal



When was the particle at rest?

(A) $t = 0$

(B) $t = 2$, $t = 8$ and $t = 14$

(C) $t = 5$ and $t = 11$

(D) $t = 8$

5) Using the substitution $u = \log_e x$, which of the following is equal to $\int_e^{e^2} \frac{1}{x\log_e x}\,dx$?

(A) $\int_e^{e^2} \frac{du}{u}$

(B) $\int_e^{e^2} \frac{du}{e^u u}$

(C) $\int_1^{2} \frac{du}{u}$

(D) $\int_1^{2} \frac{du}{e^u u}$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x} \cdot dx$

$\Rightarrow \int \frac{1}{u} \cdot du$

now $x = e^2 \Rightarrow u = \ln e^2 = 2\ln e = 2$

$x = e \Rightarrow u = \ln e = 1$

6) The acceleration of a particle is given by $a = 6x^2 - 4x - 3$, where x is the displacement in cm. The particle initially is at the origin and has a to ~~the origin~~ velocity of 3 cm/s. What is the velocity when the particle is ~~3 cm from the origin~~?

$x = 3$?

(A) $2\sqrt{7}$ cm/s

(B) $3\sqrt{7}$ cm/s

(C) $\sqrt{41}$ cm/s

(D) $\sqrt{57}$ cm/s

8 (a) $\int 2\sin^2 x\,dx - \int x^2\,dx$

very well answered by nearly all students but some did differentiate and not integrate.

3 unit 2016

$$= 2\left(\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right) - \frac{x^3}{3} + C$$

$$= x - \tfrac{1}{2}\sin 2x - \frac{x^3}{3} + C$$

(2)

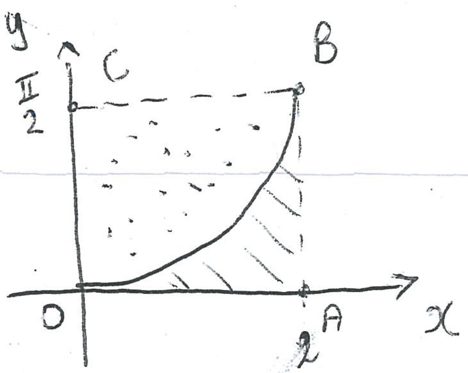
(b) $\frac{d}{dx}\left(\underset{u}{x}\,\underset{v}{\cos^{-1}} x - \sqrt{1-x^2}\right)$

Well answered using product and 'chain' rule.

$$= x \cdot \frac{-1}{\sqrt{1-x^2}} + \cos^{-1}x \times 1 - \tfrac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x$$

$$= \cancel{\frac{-x}{\sqrt{1-x^2}}} + \cos^{-1}x + \cancel{\frac{x}{\sqrt{1-x^2}}} = \cos^{-1}x = RHS$$

(2)

(c) 

$y = \sin^{-1}\frac{x}{2}$

$\sin y = \frac{x}{2}$

$x = 2\sin y$

area OABC $= 2 \times \frac{\pi}{2} = \pi$

area dotted part? $\Rightarrow \int_0^{\frac{\pi}{2}} 2\sin y\,dy$.

$$\therefore \int_0^2 \sin^{-1}\left(\frac{x}{2}\right)dx = \pi - 2\int_0^{\frac{\pi}{2}} \sin y\,dy.$$

(2)

reasonably answered. Many students are reluctant to show full working and process. The question did ask to explain!

8 (d) $f(x) = 1 + \dfrac{2}{x-3} \qquad x > 3.$

(i) Let $y = 1 + \dfrac{2}{x-3}$

$$x = 1 + \frac{2}{y-3} \qquad y > 3$$

$$\frac{2}{y-3} = \frac{x-1}{1}$$

$$\frac{y-3}{2} = \frac{1}{x-1}$$

$$y - 3 = \frac{2}{x-1}$$

$$y = f^{-1}(x) = \frac{2}{x-1} + 3 = \frac{3x-1}{x-1} \quad (1)$$

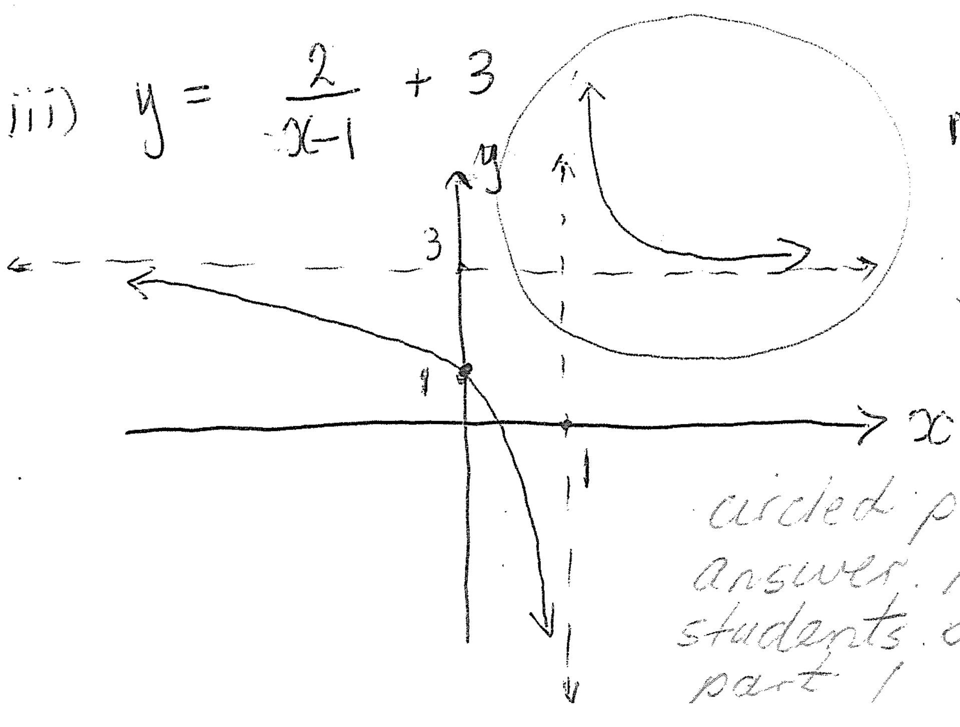
Well answered

(ii) Domain: $x > 1$ (1)

Range: $y > 3$ (1)

Badly answered by many

(iii) $y = \dfrac{2}{x-1} + 3$

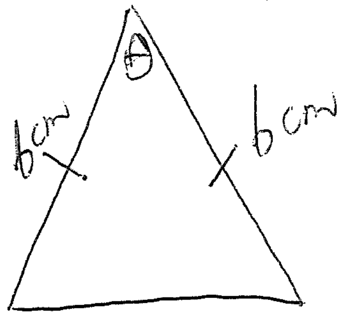


required diagram because $y > 3$, $x > 1$

(1)

circled part is the answer. Not many students only put this part!

8 (e)



$$\frac{d\theta}{dt} = 0.05 \text{ rad/s.}$$

$$A = \frac{1}{2} \times 6 \times 6 \times \sin\theta$$

$$\frac{dA}{dt} = 18 . \cos\theta . \frac{d\theta}{dt}$$

$$= 18 \times \cos\frac{\pi}{6} \times 0.05$$

$$= \cancel{18}^{9} \times \frac{\sqrt{3}}{\cancel{2}} \times \frac{5}{100}$$

$$= 0.45\sqrt{3} \text{ cm/s.}$$

3

$$\left(= \frac{9\sqrt{3}}{20} \right)$$

This was well answered! An exact answer was what the marker was looking for.

(f) $a = 3x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3x^2$

$\frac{1}{2}v^2 = \int 3x^2\, dx$

$\frac{1}{2}v^2 = x^3 + C_1$

data
$v = -\sqrt{2}$
$x = 1$
$t = 0$

$1 = 1 + C_1$

$C_1 = 0$

$\frac{1}{2}v^2 = x^3$

$v^2 = 2x^3$ (1) Up to here well answered.

$v = -\sqrt{2x^3}$

$\frac{dx}{dt} = -\sqrt{2x^3}$

$\frac{dt}{dx} = -\frac{1}{\sqrt{2x^3}} = -\frac{1}{\sqrt{2}} \cdot x^{-\frac{3}{2}}$

$t = \int -\frac{1}{\sqrt{2}} \cdot x^{-\frac{3}{2}}\, dx$

$t = -\frac{1}{\sqrt{2}} \cdot \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + C_2$ (1)

students got lost here
Problems with the integration!

$t = \frac{2}{\sqrt{2}\sqrt{x}} + C_2$

$t = \frac{\sqrt{2}}{\sqrt{x}} + C_2$

data
$t = 0$
$x = 1$

$0 = \frac{\sqrt{2}}{1} + C_2 \Rightarrow C_2 = -\sqrt{2}$

So $t = \frac{\sqrt{2}}{\sqrt{x}} - \sqrt{2}$

$(t + \sqrt{2}) = \frac{\sqrt{2}}{\sqrt{x}}$

$\sqrt{x} = \frac{\sqrt{2}}{(t+\sqrt{2})}$

$x = \frac{2}{(t+\sqrt{2})^2}$ (1)

(3)

Year 12 MX1

Q9.

(a)(i) $RHS = u^2 - u + 1 - \frac{1}{u+1}$

$= \frac{u^2(u+1) - u(u+1) + u + 1 - 1}{u+1}$

$= \frac{u^3 + u^2 - u^2 - u + u + 1 - 1}{u+1}$

$= \frac{u^3}{u+1} = LHS$ ✓ 1

(ii) Show $\int_0^4 \frac{x}{1+\sqrt{x}}\,dx = \frac{16}{3} - \ln 9$

Let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow dx = 2\sqrt{x}\,du$. ✓

When $x = 0$, $u = 0$

$x = 4$, $u = 2$

$\therefore I = \int_0^2 \frac{u^2}{1+u} \times 2\sqrt{x}\,du$

$= 2\int_0^2 \frac{u^2}{1+u} \times u\,du = 2\int_0^2 \frac{u^3}{1+u}\,du$ ✓

$= 2\int_0^2 \left(u^2 - u + 1 - \frac{1}{u+1}\right)du$ from (i)

$= 2\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln(u+1)\right]_0^2$ ✓

$= 2\left[\left(\frac{8}{3} - \frac{4}{2} + 2 - \ln 3\right) - \left(0 - 0 + 0 - \underset{0}{\ln 1}\right)\right]$ ✓

$I = \frac{16}{3} - \ln 9$. # 3

Part (i) was done well. Part (ii) was generally done well but many students did not know what to do with the square root of x in the integrand.

9(b)(i) $\frac{d(\cos\theta)}{d\theta} = -\sin\theta$

$y = \sec\theta = \frac{1}{\cos\theta} = (\cos\theta)^{-1}$

$y' = -1(\cos\theta)^{-2}(-\sin\theta)$

$= \frac{\sin\theta}{\cos^2\theta}$

$y' = \tan\theta\sec\theta$. ✓ 1

(ii) $x = \sec\theta - 1$

$I = \int_{\sqrt{2}-1}^{1} \frac{dx}{(x+1)\sqrt{x^2+2x}}$

Let $x = \sec\theta - 1$

$dx = \tan\theta\sec\theta\,d\theta$

$\Rightarrow x + 1 = \sec\theta$

and $x(x+2) = (\sec\theta - 1)(\sec\theta + 1)$

$= \sec^2\theta - 1$

$= \tan^2\theta$. ✓✓

When $x = 1$, $\sec\theta = 2$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

When $x = \sqrt{2} - 1$, $\sec\theta = \sqrt{2}$

$\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

Then $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cancel{\tan\theta}\,\cancel{\sec\theta}\,d\theta}{\cancel{\sec\theta}\,\cancel{\tan\theta}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\theta$ ✓ 3

$= \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$

Part (i) done well

Part (ii) : Many students did not use the Pythagorean identity for sec^2 -1. Also some had trouble with the bounds.

9 (c) (i) $\ddot{x} = 4x^3 - 16x$ (When $x=0$, $t=0$, $v=V_0$)

$\Rightarrow \dfrac{d(\frac{1}{2}v^2)}{dx} = 4x^3 - 16x$

integrate $\Rightarrow$ $\frac{1}{2}v^2 = x^4 - 8x^2 + C$

$v^2 = 2x^4 - 16x^2 + C$

When $x=0$, $v=V_0$ $\Rightarrow$ $V_0^2 = C$

$\therefore$ $v^2 = 2x^4 - 16x^2 + V_0^2$ (1)

Sub (1) in E where $E = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - x^4 + 8x^2$

$\Rightarrow E = \frac{1}{2}(2x^4 - 16x^2 + V_0^2) - x^4 + 8x^2$ 1

$E = \frac{1}{2}V_0^2$

$\therefore$ E is a constant and not dependent on time.

(ii) $V_0 = \sqrt{\frac{31}{8}}$

Then $E = \frac{1}{2} \times \frac{31}{8} = \frac{31}{16}$ (initial value of E) 1

In (i) many students forgot about the constant of integration.
In (ii) a common error was to let E=V0.
In (iii) most who got the quadratic equation did not check the solutions. Half a mark was given for knowing that v=0.

9 (iii) Range of particle. Particle travels along X-axis $\therefore$ need range of x values.

$v^2 = 2x^4 - 16x^2 + \frac{31}{8}$

Particle changes direction when $v=0$.

Also, if $\ddot{x} < 0$, particle is moving to left at this point

if $\ddot{x} > 0$, particle is moving to right at this point

Set $v=0 \Rightarrow 2x^4 - 16x^2 + \frac{31}{8} = 0$

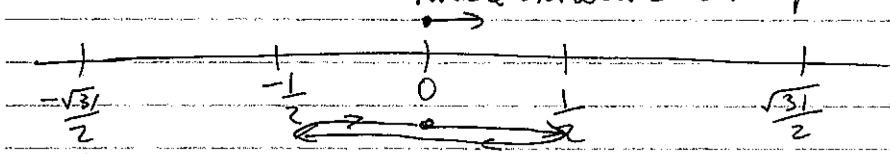
$x^2 = \dfrac{16 \pm \sqrt{256 - 8 \times \frac{31}{8}}}{4}$

$= \dfrac{16 \pm 15}{4}$

$x^2 = \frac{31}{4}$ or $\frac{1}{4}$

$\Rightarrow x = \pm\frac{\sqrt{31}}{2}$ or $\pm\frac{1}{2}$. Check these points

Particle starts at zero with positive v.



2

$x = \frac{1}{2}$, $\ddot{x} = \frac{1}{2} - 8 < 0 \Rightarrow$ particle moving to left

$x = -\frac{1}{2}$, $\ddot{x} = -\frac{1}{2} + 8 > 0 \Rightarrow$ particle moving to right

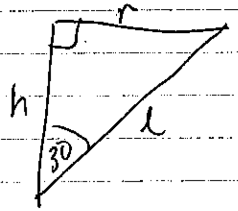
$\therefore$ particle moves between $-\frac{1}{2}$ and $\frac{1}{2}$

ie $-\frac{1}{2} \leq x \leq \frac{1}{2}$

9(d)

(i) $\frac{dV}{dt} = -3\,cm^3/s$

$\frac{dr}{dt} = ?$ when $h=5$



$h = \sqrt{l^2 - r^2}$ (1)

$\sin 30 = \frac{r}{l} = \frac{1}{2}$

$\Rightarrow l = 2r$ (2)

sub (2) in (1) $\Rightarrow h = \sqrt{4r^2 - r^2}$

$\underline{h = r\sqrt{3}}$

Now $V = \frac{1}{3}\pi r^2 h$ (cone)

$= \frac{1}{3}\pi r^2 (r\sqrt{3})$

$V = \frac{\pi}{\sqrt{3}} r^3$

$\underline{\frac{dV}{dr} = \sqrt{3}\,\pi r^2}$

Q 9(d)

(i)(cont) Then $\frac{dV}{dr} = \frac{dV}{dt}\frac{dt}{dr}$

$\sqrt{3}\,\pi r^2 = -3 \times \frac{dt}{dr}$

$\frac{dt}{dr} = \frac{\pi r^2}{\sqrt{3}}$

$\Rightarrow \underline{\frac{dr}{dt} = \frac{\sqrt{3}}{\pi r^2}}$

When $h=5$, $r = \frac{5}{\sqrt{3}}$

$\Rightarrow \frac{dr}{dt} = \frac{\sqrt{3}}{\pi \times \frac{25}{3}}$

$= \underline{\frac{-3\sqrt{3}}{25\pi}\,cm/s.}$ $= \underline{-0.0662}$ to 3sf.

3

d)(ii) $S = \pi r l$

$= \pi r (2r)$

$= 2\pi r^2$

So $\frac{dS}{dr} = 4\pi r$

Then $\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}$

$= 4\pi r \times \frac{\sqrt{3}}{\pi r^2}$

$\frac{dS}{dt} = \frac{-4\sqrt{3}}{r}\,cm^2/s$ (since S is decreasing)

2

9. (iii) When $h = 0.81\,cm$, $V_{filter} = V_{cylinder}$

$$\Rightarrow \frac{\pi}{\sqrt{3}} r^3 = \pi(10^2)h$$
$$= \pi \times 100 \times 0.81$$
$$\frac{\pi}{\sqrt{3}} r^3 = 81\pi$$
$$\Rightarrow r^3 = 81\sqrt{3} = 3^{\frac{9}{2}}$$
$$\Rightarrow r = \left(3^{9/2}\right)^{\frac{1}{3}}$$
$$r = 3^{3/2} = 3\sqrt{3}$$

$\frac{ds}{dt}$? when $h = 0.81$

$$\frac{ds}{dt} = \frac{-4\sqrt{3}}{r}$$
$$= -4\sqrt{3} \times \frac{1}{3\sqrt{3}}$$
$$= \frac{-4}{3}\ cm^2/s \quad \#$$

2

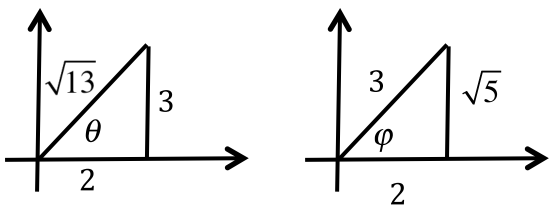
The most common error in (d) was that students were trying to differentiate a formula with respect to two variables. The other common mistake was to substitute in h-=5 at the start of the problem.

SHS Maths Ext 1 Task 3 2016

Question 10

(a) $E = \sin\left[\tan^{-1}\left(\tfrac{3}{2}\right) + \cos^{-1}\left(\tfrac{2}{3}\right)\right]$

Let $\theta = \tan^{-1}\left(\tfrac{3}{2}\right)$ and $\varphi = \cos^{-1}\left(\tfrac{2}{3}\right)$



$$\begin{aligned} E &= \sin\theta\cos\varphi + \cos\theta\sin\varphi \\ &= \frac{3}{\sqrt{13}}.\frac{2}{3} + \frac{2}{\sqrt{13}}.\frac{\sqrt{5}}{3} \\ &= \frac{6+2\sqrt{5}}{3\sqrt{13}} \end{aligned}$$

[This part was generally well-answered. Some unnecessarily rationalized the denominator.]

(b) (i) The volume will be found by rotation of a circular arc about the y-axis. The equation of the circle is

$$\begin{aligned} x^2 + (y-10)^2 &= 100 \\ x^2 &= 100 - (y-10)^2 \\ x^2 &= 20y - y^2 \end{aligned}$$

Thus
$$\begin{aligned} V &= \pi\int_0^h \left(20y - y^2\right)dy \\ &= \pi\left[10y^2 - \frac{y^3}{3}\right]_0^h \\ &= \pi\left(10h^2 - \frac{h^3}{3}\right) \end{aligned}$$

[Those who realised that integration was required, generally answered well. The others scored poorly.]

(ii) By the Chain Rule $\dfrac{dh}{dt}=\dfrac{dh}{dV}\dfrac{dV}{dt}$

$$\text{Now}\quad \frac{dV}{dt}=20\text{ cm}^3/\text{min}$$

$$\frac{dV}{dh}=\pi\left(20h-h^2\right)$$

$$\frac{dh}{dV}=\frac{1}{\pi\left(20h-h^2\right)}$$

$$\text{Thus at } h=5\quad \frac{dh}{dt}=\frac{1}{\pi\left(20\times5-5^2\right)}\times20$$

$$=\frac{20}{75\pi}$$

$$\approx 0.08\text{ cm/min}$$

[This was well-answered, even by those who failed in part (i). Many failed to give the approximate answer, but were not penalised.]

(c) $P_n:\ \displaystyle\sum_{r=1}^{n} r\ln\left(\frac{r+1}{r}\right)=\ln\left(\frac{(n+1)^n}{n!}\right)$

$P_1:\ LHS=1.\ln\left(\dfrac{2}{1}\right)$ $\qquad RHS=\ln\left(\dfrac{2^1}{1!}\right)$

$=\ln2 \qquad\qquad =\ln2$

$LHS=RHS,\quad \therefore P_1$ is true.

$P_k:$ Assume the proposition is true for $n=k,\ k\in J^+,\ 1\le k\le n$

That is $\displaystyle\sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right)=\ln\left(\frac{(k+1)^k}{k!}\right)$

$P_{k+1}:$ Required to prove $P_k\to P_{k+1}$, that is

$$\sum_{r=1}^{k+1} r\ln\left(\frac{r+1}{r}\right)=\ln\left(\frac{(k+2)^{k+1}}{(k+1)!}\right)$$

$$LHS=\sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right)+(k+1)\ln\left(\frac{(k+2)}{k+1}\right)$$

$$=\ln\left(\frac{(k+1)^k}{k!}\right)+(k+1)\ln\left(\frac{(k+2)}{k+1}\right)\text{ by our assumption}$$

$$=\ln\left(\frac{(k+1)^k}{k!}\right)+\ln\left(\frac{(k+2)}{k+1}\right)^{k+1}\text{ (properties of logs)}$$

$$= \ln\left(\frac{(k+1)^k}{k!} \times \frac{(k+2)^{k+1}}{(k+1)^{k+1}}\right)$$

$$= \ln\left(\frac{(k+2)^{k+1}}{(k+1)!}\right)$$

$= RHS$ as required

Hence by the Principle of Mathematical Induction, P_n is true for positive integral n.

[Many failed to show that $P_k \to P_{k+1}$, simply leaving out the steps they could not fathom.]

(d) (i) Clearly $\theta = \angle OPB - \angle OPA$

$$= \tan^{-1}\left(\frac{4}{x}\right) - \tan^{-1}\left(\frac{1}{x}\right)$$

[Most were successful in this part.]

(ii) $$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{4}{x}\right)^2} \times \left(-4x^{-2}\right) - \frac{1}{1+\left(\frac{1}{x}\right)^2} \times \left(-x^{-2}\right)$$

$$= \frac{-4}{x^2+16} + \frac{1}{x^2+1}$$

$$= \frac{12-3x^2}{\left(x^2+16\right)\left(x^2+1\right)}$$

Stationary points when $12 - 3x^2 = 0$

$x = \pm 2$

When $x = 2$:

x	1	2	3
θ'	$\frac{9}{34}$	0	$\frac{-15}{250}$
	rising	stat	falling

Thus θ is a maximum when $x = 2$.

[Most candidates were able to handle the rather algebraically challenging derivative. Quite a few failed to determine the nature of the turning point. Some who did lost marks for not using numbers in the derivative row of the table above.]

(iii) Maximum angle is $\theta = \tan^{-1} 2 - \tan^{-1} \frac{1}{2}$.

$$\text{Now } \tan\theta = \frac{2 - \frac{1}{2}}{1 + 2 \times \frac{1}{2}}$$
$$= \frac{3}{4}$$

$$\therefore \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

[This last part was well-answered by those (most) who attempted it.]