Sydney Technical High School

YEAR 12 ASSESSMENT TASK

JUNE 1998

3 UNIT MATHEMATICS

Time Allowed: 70 minutes

Instructions:
* Attempt all questions
* Answers to be written on the paper provided.
* Start each question on a new page.
* Marks may not be awarded for careless or badly arranged working.
* This question paper must be stapled on top of your answers.
* Marks shown are approximate and may be changed.
* A table of standard integrals is provided

Name: ______________________ Teacher: ______________________

Question 1	Question 2	Question 3	Question 4	Question 5

TOTAL

Question 1 **12 marks**

a) Evaluate $\sin^{-1}(-\frac{1}{2})$ in terms of π 2

b) Differentiate $\sin^{-1} 4x$ 2

c) (i) $\int \frac{dy}{\sqrt{100-y^2}}$ 1

(ii) $\int \frac{dt}{8+t^2}$ 2

d) The rate of change of y with respect to t is given by 2

$\frac{dy}{dt} = (2t-1)^9$ If $y = 0$ when $t = \frac{1}{2}$ find y when $t = 1$

e) State the domain and range of $y = 4\cos^{-1}\frac{x}{3}$ 3

Question 2 **12 marks**

a) By letting $A = \sin^{-1}\frac{3}{8}$ find the exact value of 2

$\cos\left\{\sin^{-1}\frac{3}{8}\right\}$

b) (i) For the function $f(x) = x^2 - 2$ state the largest possible domain for which 1

$f(x)$ has an inverse $f^{-1}(x)$, containing the point $x = 1$

(ii) Find the equation of the inverse function $f^{-1}(x)$ 1

(iii) Sketch $f(x)$ for the restricted domain in part (i) and $f^{-1}(x)$ on the same axes 2

(iv) Find the point(s) of intersection of $f(x)$ and $f^{-1}(x)$ on your sketch in part (iii) 2

c) A particle moves in a straight line so that its velocity (m/s) is given by $v = \frac{1}{1+t}$

(i) If the particle was initially at rest at the origin find displacement (x) as a function of time (t) 2

(ii) Show the particle is moving away from the origin as t increases. 1

(iii) Find the acceleration of the particle when $t = 0$ 1

Question 3 12 marks

a) Find $\int \frac{dx}{\sqrt{9-4x^2}}$ 2

b) (i) Sketch $y = \sin^{-1} x$ 2

(ii) Find the area bounded by $y = \sin^{-1} x$, the x axis and $x = 1$ in exact form. 3

c) V_P and V_Q are the velocities of two particles 5

P and Q moving along with the x axis at time t (seconds)

$$V_P = \frac{1}{t^2+4} \qquad V_Q = \frac{t}{t^2+4}$$

Assuming each particle was initially at the origin, which particle has moved the greater distance after one second?

(Show full working to support your answer)

Question 4 12 marks

a) (i) Write an expression for $\tan 2A$ 1

(ii) By letting $A = \tan^{-1}\frac{1}{4}$ and using the above identity prove 2

$$2\tan^{-1}\frac{1}{4} = \tan^{-1}\frac{8}{15}$$

b) A raindrop falls so that the rate of change of velocity with respect to time is given by

$$\frac{dv}{dt} = \frac{1}{2}(2g - V)$$

where g is acceleration due to gravity (a constant), V is velocity in metres per second and t is time in seconds.

(i) Show that $V = 2g + Ae^{-\frac{1}{2}t}$, where A is a constant, is a solution to the above differential equation 2

(ii) Find the limiting velocity of the raindrop in terms of g (ie at $t \to \infty$) 1

(iii) If the raindrop falls from rest find A and the velocity after 6 seconds (both answers in terms of g) 2

(iv) Find the time for the raindrops to reach a velocity of $\frac{1}{2}g$ m/s. 1

c) Find $\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$ using the substitution $x = u^2$ 3

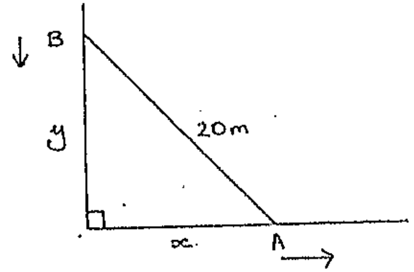
Question 5 12 marks

a) Find the general solution for $\sin\theta = \frac{1}{\sqrt{2}}$ 3

b) (i) Write $x^2 + 6x + 10$ in the form $(x+a)^2 + b$ 1

(ii) Hence find $\int \frac{dx}{x^2+6x+10}$ 2

c) The top B of ladder 20m long rests against a wall and the foot A is on level ground. If the foot is slipping away from the wall at a constant rate of 1 m per 3 seconds, find the rate at which the top is descending at the instant when A is 12 m from the wall 6



(use x as the horizontal distance and y the vertical distance as shown above).

STANDARD INTEGRALS

$\int x^n\,dx$	$= \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,$ if $n < 0$
$\int \frac{1}{x}\,dx$	$= \ln x,\ x > 0$
$\int e^{ax}\,dx$	$= \frac{1}{a}e^{ax},\ a \neq 0$
$\int \cos ax\,dx$	$= \frac{1}{a}\sin ax,\ a \neq 0$
$\int \sin ax\,dx$	$= -\frac{1}{a}\cos ax,\ a \neq 0$
$\int \sec^2 ax\,dx$	$= \frac{1}{a}\tan ax,\ a \neq 0$
$\int \sec ax \tan ax\,dx$	$= \frac{1}{a}\sec ax,\ a \neq 0$
$\int \frac{1}{a^2+x^2}\,dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$= \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$= \ln\left(x + \sqrt{x^2-a^2}\right),\ x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$= \ln\left(x + \sqrt{x^2+a^2}\right)$

3UNIT ASSES TASK 3 JUNE 1998

QUESTION 1

a) $\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\frac{1}{2} = -\frac{\pi}{6}$

b) $\frac{d}{dx}\left(\sin^{-1}4x\right) = \frac{4}{\sqrt{1-16x^2}}$

c) i) $\int\frac{dy}{\sqrt{100-y^2}} = \sin^{-1}\frac{y}{10} + c$

ii) $\int\frac{dt}{8+t^2} = \frac{dt}{(2\sqrt{2})^2+t^2} = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{t}{2\sqrt{2}}+c$

d) $\frac{dy}{dt} = (2t-1)^9$

$y = \frac{(2t-1)^{10}}{20} + c$ sub $y=0$, $t=\frac{1}{2}$

$\therefore 0 = c$

$y = \frac{(2t-1)^{10}}{20}$ sub $t=1$

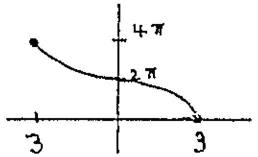
$y = \frac{1}{20}$

e) $y = 4\cos^{-1}\frac{x}{3}$

$\frac{y}{4} = \cos^{-1}\frac{x}{3}$

$-1 \le \frac{x}{3} \le 1$ $\quad 0 \le \frac{y}{4} \le \pi$

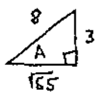
$\therefore$ D: $-3 \le x \le 3$ $\quad$ R: $0 \le y \le 4\pi$



QUESTION 2

a) $A = \sin^{-1}\frac{3}{8}$

$\therefore \sin A = \frac{3}{8}$



$\therefore \cos\left\{\sin^{-1}\frac{3}{8}\right\} = \cos A = \frac{\sqrt{55}}{8}$

b) $f(x) = x^2 - 2$

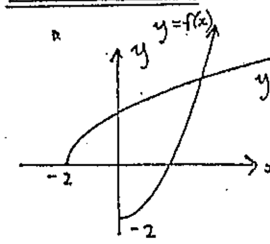
i) Domain : $x \ge 0$

ii) $x = y^2 - 2$.

$x + 2 = y^2$

$\therefore y = \sqrt{x+2}$ $\quad$ since domain of $f(x)$ is $x \ge 0$

iii)



iv) pt intersection on line $y = x$

$x = x^2 - 2$

$0 = x^2 - x - 2$

$(x-2)(x+1) = 0$

$\therefore$ pt $(2,2)$ only in above domain

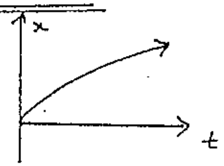
c) $v = \frac{dx}{dt} = \frac{1}{1+t}$

i) $x = \ln(1+t) + c$ sub $x=0$ $t=0$ $\therefore c = 0$

$\therefore x = \ln(1+t)$

ii) $t \to \infty$ $\quad x \to \infty$

or by sketch



iii) $\frac{d^2x}{dt^2} = \frac{-1}{(1+t)^2}$

$\therefore$ when $t = 0$ $\quad$ accel^n = -1 m/s^2

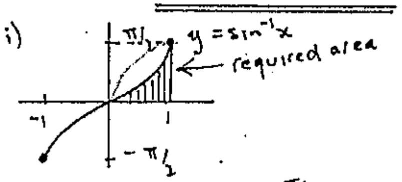
QUESTION 3

a) $\int\frac{dx}{\sqrt{9-4x^2}} = \int\frac{dx}{\sqrt{4\left(\frac{9}{4}-x^2\right)}}$

$= \frac{1}{2}\int\frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2-x^2}}$

$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$

b) i)



ii) A = area rectangle $- \int_0^{\pi/2}\sin y\,dy$

$= \frac{\pi}{2} - \left[-\cos y\right]_0^{\pi/2}$

$= \frac{\pi}{2} - \left[-\cos\frac{\pi}{2} - -\cos 0\right]$

$= \left(\frac{\pi}{2} - 1\right)$ unit^2

c) $V_P = \frac{1}{t^2+4}$ $\quad x_P = \int\frac{1}{t^2+4}\,dt$

$\therefore x_P = \frac{1}{2}\tan^{-1}\frac{t}{2} + c$ $\quad$ sub $x=0$, $t=0$

$\therefore c = 0$

$x_P = \frac{1}{2}\tan^{-1}\frac{t}{2}$

$\therefore t = 1$ $\quad x_P = \frac{1}{2}\tan^{-1}\frac{1}{2} = \cdot 232$ units

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$V_Q = \frac{t}{t^2+4}$

$x_Q = \frac{1}{2}\int\frac{2t}{t^2+4}\,dt$

$= \frac{1}{2}\ln(t^2+4) + c$

sub $x=0$, $t=0$ $\therefore c = -\frac{1}{2}\ln 4$

$\therefore x_Q = \frac{1}{2}\ln(t^2+4) - \frac{1}{2}\ln 4$

$\therefore t = 1$ $\quad x_Q = \frac{1}{2}\ln 5 - \frac{1}{2}\ln 4 = \frac{1}{2}\ln\frac{5}{4}$

QUESTION 4

a) i) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

ii) $A = \tan^{-1}\frac{1}{4}$

$\therefore \tan A = \frac{1}{4}$

$\therefore \tan 2A = \frac{2\times\frac{1}{4}}{1-\frac{1}{16}} = \frac{8}{15}$

$\therefore 2A = \tan^{-1}\frac{8}{15}$

$\therefore 2\tan^{-1}\frac{1}{4} = \tan^{-1}\frac{8}{15}$

b) i) $V = 2g + Ae^{-\frac{1}{2}t}$

$\therefore \frac{dV}{dt} = -\frac{1}{2}Ae^{-\frac{1}{2}t}$

since $-Ae^{-\frac{1}{2}t} = 2g - V$

$\therefore \frac{dV}{dt} = \frac{1}{2}(2g - V)$

ii) as $t \to \infty$

$V = 2g + \cancel{\frac{A}{e^{\frac{1}{2}t}}} \Rightarrow 0$

$\therefore V = 2g$ limiting velocity m/s

iii) $t = 0$ $\quad V = 0$

$\therefore 0 = 2g + A$

$\therefore A = -2g$

$\therefore V = 2g - 2ge^{-\frac{1}{2}t}$

sub $t = 6$

$V = 2g - 2ge^{-\frac{1}{2}\times 6}$

$= 2g - 2ge^{-3}$ (exact)

OR $V \doteqdot 1\cdot 9g$ m/s

iv) $\frac{1}{2}\cancel{g} = 2\cancel{g} - 2\cancel{g}e^{-\frac{1}{2}t}$

$\frac{1}{2} = 2 - 2e^{-\frac{1}{2}t}$

$t = \frac{\ln 3/4}{-\frac{1}{2}}$

c) $x = u^2$

$\frac{dx}{du} = 2u \quad \therefore dx = 2u \cdot du$

$\int \frac{dx}{\sqrt{x}\sqrt{1-x}} = \int \frac{2\cancel{u} \cdot du}{\cancel{u}\sqrt{1-u^2}}$

$= 2\int \frac{du}{\sqrt{1-u^2}}$

$= 2 \sin^{-1} u + c$

$= 2 \sin^{-1} \sqrt{x} + c$

QUESTION 5

a) $\sin \theta = \frac{1}{\sqrt{2}}$

$\therefore \theta = n\pi + (-1)^n \sin^{-1} 1/\sqrt{2}$

$\theta = n\pi + (-1)^n \frac{\pi}{4}$

where n is an integer

b) i) $x^2+6x+10 = x^2+6x+9+1$

$= (x+3)^2+1$

ii) $\int \frac{dx}{x^2+6x+10} = \int \frac{1}{(x+3)^2+1}$

$= \tan^{-1}(x+3) + c$

c) $\frac{dx}{dt} = \frac{1}{3}$ m/s

want $\frac{dy}{dt}$ when $x = 12$

$20^2 = x^2 + y^2$

$y = \sqrt{400-x^2}$

$\therefore \frac{dy}{dx} = \frac{1}{2} \cdot 2x \, (400-x^2)^{-1/2}$

$= \frac{-x}{\sqrt{400-x^2}}$

$\therefore \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$\therefore \frac{dy}{dt} = \frac{-x}{\sqrt{400-x^2}} \cdot \frac{1}{3}$

sub $x = 12$

$= \frac{-12}{\sqrt{400-144}} \times \frac{1}{3}$

$= \frac{-12}{16} \times \frac{1}{3}$

$\frac{dy}{dt} = -\frac{1}{4}$

$\therefore$ B is descending at $\frac{1}{4}$ m/s