Name:______________________________

Teacher/Class:______________________________

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12

HSC ASSESSMENT TASK 3

2005

EXTENSION 1 MATHEMATICS

Time Allowed: **70 minutes**

Instructions:

- Write your name and class at the top of each page.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start each question on a new page.

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total
/10	/10	/11	/10	/11	/10	/62

Question 1 (10 marks)

a) Differentiate

(i) xe^{x^2}

(ii) $\log_{10} x$ (4)

b) Find

(i) $\int 4e^{2x+1}\,dx$

(ii) $\int \tan^2\theta\,d\theta$ (3)

c) Show that $d/dx(\sin^3 x) = 3\cos x - 3\cos^3 x$ and hence, or otherwise find

$\int (\cos x - \cos^3 x)dx$ (3)

Question 2 (10 marks)

a) Find the exact value of (3)

(i) $\log_2 \sqrt[3]{4}$

(ii) $\sin^{-1}(-\frac{1}{\sqrt{2}})$

b) State the domain and range of $y = 4\sin^{-1}(1-x)$ (2)

c) Find $\int \sin^2 4x\,dx$ (2)

d) Find $\int \frac{4}{\sqrt{9-4x^2}}\,dx$ (3)

Question 3 (11 marks)

a) Differentiate (4)

(i) $\tan^{-1}(\frac{x}{2})$

(ii) $\sin^{-1}(x^2)$

b) Solve $\log_8 2 = \log_x 5$ (2)

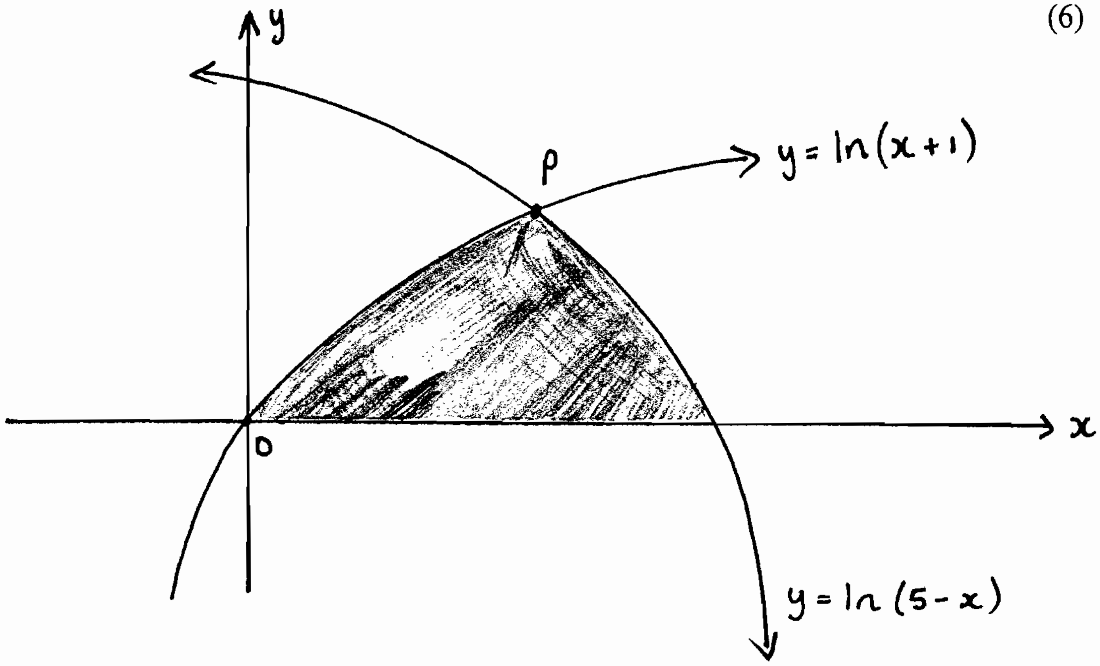
c) (i) Express $\sqrt{3}\sin x - \cos x$ in the form of $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ (5)

(ii) Sketch the graph of $y = \sqrt{3}\sin x - \cos x$ for $0 \le x \le 2\pi$, showing intercepts and endpoints.

(iii) Use your sketch to state the **number** of solutions to the equation $\sqrt{3}\sin x - \cos x = -1$.

Question 4 (10 marks)

a) (6)



The diagram shows the graphs of $y = \ln(x+1)$ and $y = \ln(5-x)$ intersecting at **P**.

(i) Show that the coordinates of **P** are (2, ln 3)

(ii) The area, **A**, enclosed by the curve y = ln (x + 1), the **y axis** and the lines y = 0 and y = ln3 is given by the integral,

$$A = \int_0^{\ln 3} \left(e^y - 1\right)\ dy$$

Use this integral to calculate the exact area, **A**.

(iii) Calculate the exact area of the shaded region shown above.

b) (i) Sketch $y = \cos^{-1} x$ showing all intercepts and end points (3)

(ii) Hence find $\int_{-1}^{1} \cos^{-1} x\ dx$

c) Write down the general solution for $\tan x = \dfrac{1}{\sqrt{3}}$ (1)

Question 5 **(11 marks)**

a) Given $\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta = \frac{\pi+2}{8}$ (3)

use the substitution $x = 2\sin\theta$, to evaluate $\int_0^{\sqrt{2}} \sqrt{4-x^2} \ \mathrm{dx}$

b) $y = \sin x + \sin x \cos x$ for $0 \le x \le 2\pi$ (8)

(i) Show that $\frac{dy}{dx} = \cos x + \cos 2x$

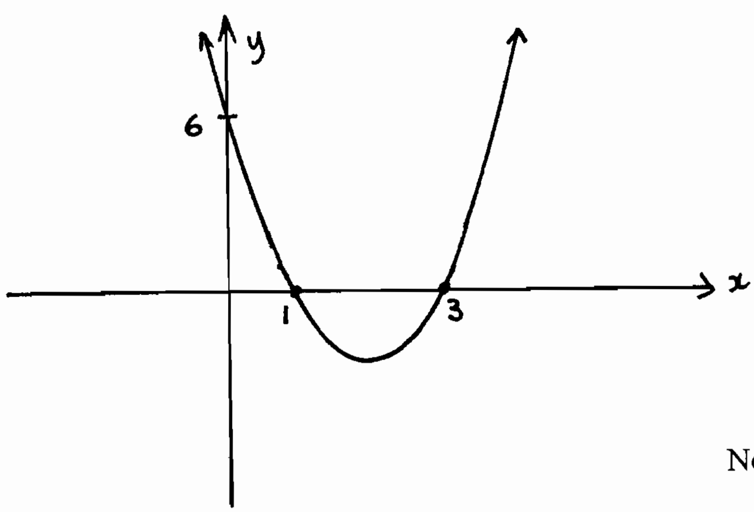
(ii) Find the co ordinates of the three stationary points and determine their nature.

(iii) Sketch the curve for the given domain.

Question 6 **(10 marks)**

a) Find $\int \frac{x+1}{x^2+4} \, dx$ (3)

b) (7)



Not to scale

(i) Show that the equation of the parabola $y = f(x)$ is given by $y = 2x^2 - 8x + 6$ and state the co ordinates of the vertex

(ii) State the largest possible domain, that includes $x = 0$, for which $y = f(x)$ has an inverse.

(iii) State the domain of $y = f^{-1}(x)$.

(iv) What is the equation of the inverse function $y = f^{-1}(x)$.

END OF TEST

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Solutions 3u/ext 1 Task 3

HSC 2005

Teacher's Name: ______________ **Student's Name/N°:** ______________

Question 1

a) i. $y = x e^{x^2}$

$\frac{dy}{dx} = vu' + uv'$

$= e^{x^2} . 1 + x . 2x e^{x^2}$ ✓✓

$= e^{x^2}(1 + 2x^2)$

ii. $\log_{10} x = \frac{\ln x}{\ln 10}$ ✓

$\therefore \frac{d}{dx}(\log_{10} x) = \frac{1}{x \ln 10}$ ✓

b) i. $\int 4e^{2x+1} \, dx$

$= 2e^{2x+1} + C$ ✓

ii. $\int \tan^2\theta \, d\theta$

$= \int (\sec^2\theta - 1) \, d\theta$ ✓

$= \tan\theta - \theta + C$ ✓

c) $\frac{d}{dx}(\sin^3 x)$

$= 3(\sin x)^2 . \cos x$ ✓

$= 3\cos x(1 - \cos^2 x)$

$= 3\cos x - 3\cos^3 x$ ✓

$\therefore \int \cos x - \cos^3 x \, dx$ ✓

$= \frac{1}{3}\sin^3 x + C$

Question 2 .

a) i. $\log_2 \sqrt[3]{4} = \log_2 (2^2)^{1/3}$

$= \frac{2}{3} \log_2 2$

$= \frac{2}{3}$ ✓

ii. $\sin^{-1}(-\frac{1}{\sqrt{2}}) = -\frac{\pi}{4}$ ✓✓

① = sign

b) $y = 4\sin^{-1}(1 - x)$

D: $-1 \le 1 - x \le 1$

$-2 \le -x \le 0$

$2 \ge x \ge 0$

$\therefore 0 \le x \le 2$ ✓

R: $-2\pi \le y \le 2\pi$ ✓

c) $\int \sin^2 4x \, dx$

$= \frac{1}{2}\int 1 - \cos 8x \, dx$ ✓✓

$= \frac{1}{2}\left[x - \frac{1}{8}\sin 8x\right] + C$

d) $\int \frac{4}{\sqrt{9 - 4x^2}} \, dx$

$= 4\int \frac{1}{\sqrt{4(9/4 - x^2)}} \, dx$ ✓

$= 4\int \frac{1}{2\sqrt{9/4 - x^2}}$ ✓

$= 2\sin^{-1} \frac{x}{3/2}$

$= 2\sin^{-1}\frac{2x}{3} + C$ ✓

Teacher's Name: ______________ **Student's Name/Nº:** ______________

Question 3

a) i. $\frac{d}{dx}\left(\tan^{-1}\frac{x}{2}\right) = \frac{2}{x^2+4}$ ✓

ii. $\sin^{-1}(x^2) = y$ let $u = x^2$

$y = \sin^{-1} u$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{1}{\sqrt{1-u^2}} \cdot 2x$ ✓

$= \frac{2x}{\sqrt{1-x^4}}$ ✓

b) $\log_8 2 = \log_8 8^{1/3} = 1/3$

$\therefore \log_x 5 = 1/3$ ✓

$x^{1/3} = 5$

$x = 125$ ✓

c) $R = \sqrt{\sqrt{3}^2 + 1^2}$

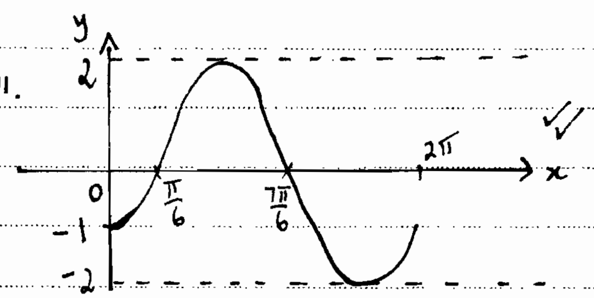
$= 2$

$\tan\alpha = 1/\sqrt{3}$ ✓✓

$\alpha = \pi/6$

$\therefore \sqrt{3}\sin x - \cos x = 2\sin(x - \frac{\pi}{6})$

ii. ✓✓



iii.

three

Question 4

a) i. $\ln(x+1) = \ln(5-x)$

$x+1 = 5-x$ ✓

$2x = 4$

$x = 2$ $y = \ln 3$

$\therefore P(2, \ln 3)$

ii. $\int_0^{\ln 3} e^y - 1 \, dy$

$= e^y - y \Big]_0^{\ln 3}$ ✓

$= e^{\ln 3} - \ln 3 - (e^0 - 0)$

$= 3 - \ln 3 - 1$

$= 2 - \ln 3$ ✓

iii. $y = \log_e(5-x) \rightarrow e^y = 5 - x$

$x = 5 - e^y$

$A_y = \int_0^{\ln 3} 5 - e^y \, dy$ ✓

$= 5y - e^y \Big]_0^{\ln 3}$

$= 5\ln 3 - 3 - (0 - e^0)$

$= 5\ln 3 - 3 + 1$

$= 5\ln 3 - 2$ ✓

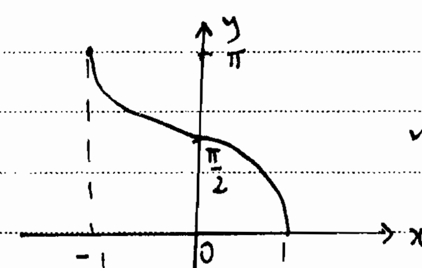
$\therefore$ shaded area

$= 5\ln 3 - 2 - (2 - \ln 3)$

$= 5\ln 3 - 2 - 2 + \ln 3$

$= 6\ln 3 - 4$ ✓

b) ✓✓



$\int^{1} \cos x \, dx = 1 \times \pi$ ✓

Question 4 con t

c) $\tan x = 1/\sqrt{3}$

$$x = n\pi + \frac{\pi}{6}$$

or ✓①

$$x = n.180^\circ + 30^\circ$$

where n is any integer ← must have

Question 5

a) $\int_0^{\sqrt{2}} \sqrt{4 - x^2}\, dx$

$x = 2\sin\theta$

$\sqrt{2} = 2\sin\theta$ $\qquad \frac{dx}{d\theta} = 2\cos\theta$

$\theta = \frac{\pi}{4}$

$0 = 2\sin\theta$

$\theta = 0$

$$\therefore \int_0^{\pi/4} \sqrt{4 - 4\sin^2\theta} \,.\, 2\cos\theta\, d\theta$$

$$= \int_0^{\pi/4} 2\sqrt{1 - \sin^2\theta} \,.\, 2\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/4} \cos\theta . \cos\theta\, d\theta$$

$$= 4\int_0^{\pi/4} \cos^2\theta\, d\theta$$

$$= 4 \times \left(\frac{\pi + 2}{8}\right)$$

$$= \frac{\pi + 2}{2}$$

b) $y = \sin x + \sin x \cos x$

i) $y = \sin x + \frac{1}{2}\sin 2x$

$$\therefore \frac{dy}{dx} = \cos x + \frac{1}{2}\cos 2x \times 2$$

$$= \cos x + \cos 2x \quad ✓$$

ii) Stat pts $\frac{dy}{dx} = 0$

$\cos x + \cos 2x = 0$

$\cos x + 2\cos^2 x - 1 = 0$ ✓

$2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$\therefore \cos x = \frac{1}{2}$ $\qquad \cos x = -1$

$\therefore x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\qquad x = \pi$ ✓

$\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{4}\right)$ MAX TP ✓ Nature

x	1^c	$\pi/3$	1.5^c
y'	0.12 /	—	-0.9 \

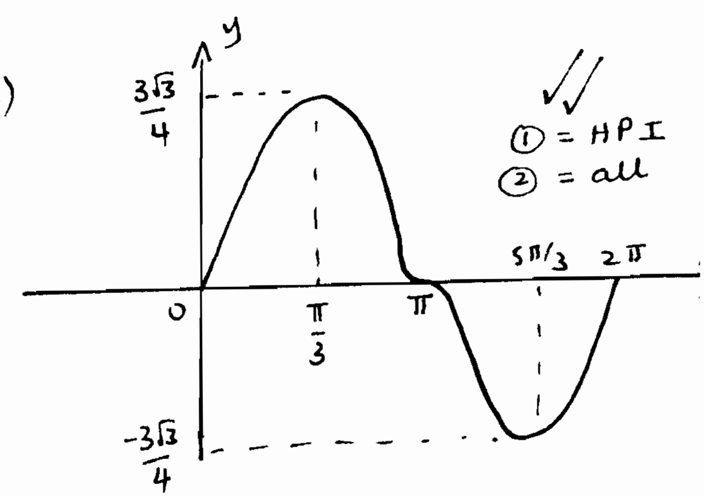
$\left(\frac{5\pi}{3}, -\frac{3\sqrt{3}}{4}\right)$ MIN TP ✓

x	5^c	$\frac{5\pi}{3}$	5.5^c
y'	-0.5 \	—	0.7 /

$(\pi, 0)$ ✓ HPI

x	3	π	3.5
y'	-0.02 \	—	-0.1 \

iii)



✓✓ ① = HPI ② = all

Question 6

a) $\int \frac{x+1}{x^2+4}\,dx$

$= \int \frac{x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$ ✓

$= \frac{1}{2}\ln(x^2+4)$ ✓ $+ \frac{1}{2}\tan^{-1}\frac{x}{2}$ ✓ $+ C$

b) $y = a(x-1)(x-3)$ thru $(0,6)$

I. $6 = a(-1)(-3)$

$a = 2$ ✓

$\therefore\ y = 2(x-1)(x-3)$

$y = 2x^2 - 8x + 6$

vertex $(2,-2)$ ✓

II. $x \leq 2$ ✓

III. $y = f^{-1}(x)$ D: $x \geqslant -2$ ✓

IV. $x = 2y^2 - 8y + 6$ ✓

$\frac{x}{2} = y^2 - 4y + 3$

$\frac{x}{2} = y^2 - 4y + 4 - 1$

$\frac{x}{2} + 1 = (y-2)^2$

$y - 2 = \pm\sqrt{\frac{x+2}{2}}$

$y = 2 \pm \sqrt{\frac{x+2}{x}}$ ✓

$y = 2 - \sqrt{\frac{x+2}{2}}$ ✓

$\therefore\ f^{-1}(x) = 2 - \sqrt{\frac{x+2}{2}}$