SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12

HSC ASSESSMENT TASK 3

JUNE 2006

MATHEMATICS
EXTENSION 1

Time Allowed: 70 Minutes

Instructions:

- Write your name and class at the top of each page.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Marks indicated are a guide only and may be varied if necessary.
- Start each question on a new page
- Standard integrals can be found on the last page.

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total
/10	/10	/10	/10	/10	/12	/62

Question 1

(a) Find the exact value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ **1**

(b) Differentiate the following

i. $y = e^{x^2}$ **1**

ii. $y = \sin^{-1} 3x$ **2**

iii. $y = \log_e \dfrac{x^2}{x-1}$ **2**

(c) Simplify $\log_a x^3 \div \log_a \sqrt{x}$ **2**

(d) Find the general solution of $2\cos x = -1$ **2**

Question 2 (Begin on the next page)

(a) Find i $\int \frac{x}{x^2+1}\,dx$ **1**

ii $\int 2x^2 e^{x^3}\,dx$ **1**

(b) Find k if $x^{k+3} = e^{7\ln x}$ where $x > 0$ **2**

(c) Consider $f(x) = \sin^{-1}(2x+1)$

i. Find the domain of $y = f(x)$ **1**

ii. Find the inverse function $y = f^{-1}(x)$ **2**

iii. State the domain and range of $y = f^{-1}(x)$ **2**

iv. Sketch $y = f^{-1}(x)$ **1**

Question 3 (Begin on the next page)

(a) Consider the function $y = x\ln x$ where $x > 0$.

i. Find the stationary point and determine its nature. **2**

ii. Show that the curve is always concave upwards. **1**

iii. Find $\lim\limits_{x \to 0} x\ln x$. **1**

iv. Where does the curve cut the x axis? **1**

v. Sketch the curve showing all important features. **2**

(b) Find $\int \frac{dx}{\sqrt{16-25x^2}}$. **3**

Question 4 (Begin on the next page)

(a) During the early summer months the rate of increase of the population, P, of fruit flies is proportional to the excess of the population over 3000

so $\frac{dP}{dt} = k(P-3000)$ where t is in months and k is a constant.

At the beginning of summer the population is 4000 and 1 month later it is 10 000.

i. Show that $P = 3000 + Ae^{kt}$ is a solution of the differential equation, where A is a constant. **1**

ii. Find the value of A and the exact value of k. **2**

iii Find to the nearest 100, the population after 2½ months. **1**

iv. After how many weeks does the population reach ½ million ? **2**

(b) A spherical balloon is being inflated and its volume increases at the constant rate of 50 mm^3 per second.

At what rate is its surface area increasing when the radius is 20mm ? **4**

Question 5 (Begin on the next page)

(a) i. Show that $\int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta = \frac{\pi+2}{8}$. **3**

ii. Hence, using the substitution $x = 2\sin\theta$ or otherwise, **3**

evaluate $\int_0^{\sqrt{2}} \sqrt{4-x^2}\, dx$

(b) Use the substitution $x = u^2, u > 0$ to find the exact value of **4**

$$\int_1^3 \frac{1}{(x+1)\sqrt{x}}\, dx$$

Question 6 (Begin on the next page)

i. Solve the equation $x^4 + x^2 - 1 = 0$, giving answers correct to two decimal places. **2**

ii. On the same axes, draw the graphs of $y = \tan^{-1}x$ and $y = \cos^{-1}x$, showing all important features. Mark the point P where the curves intersect. **2**

iii Let $\tan^{-1}x = \alpha$. Find an expression for $\cos\alpha$ in terms of x. **1**

iv. Show that, if $\tan^{-1}x = \cos^{-1}x$ then $x^4 + x^2 - 1 = 0$. **2**

v. State the coordinates of P. **1**

vi. Find the area enclosed by the curves and the y axis from

$y = \frac{\pi}{2}$ to $y = 0$ **4**

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

(a) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\tan^{-1}\frac{1}{\sqrt{3}}$

$= \frac{-\pi}{6}$

(b) i. $\frac{dy}{dx} = 2xe^{x^2}$

ii. $\frac{dy}{dx} = \frac{3}{\sqrt{1-9x^2}}$

iii. $y = \log_e x^2 - \log_e(x-1)$

$y = 2\log_e x - \log_e(x-1)$

$\frac{dy}{dx} = \frac{2}{x} - \frac{1}{x-1}$

(c) $\frac{3\log_a x}{\frac{1}{2}\log_a x} = 6$

(d) $\cos x = -\frac{1}{2}$

$x = 2n\pi \pm \frac{2\pi}{3}$ where n is an integer

Question 2

(a) i. $\frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)+c$

ii. $\frac{2}{3}\int 3x^2 e^{x^3}\,dx = \frac{2}{3}e^{x^3}+c$

(b) $x^{k+3} = e^{7\ln x}$

$x^{k+3} = e^{\ln x^7}$

(c) $f(x) = \sin^{-1}(2x+1)$

i. $-1 \le 2x+1 \le 1$

$-2 \le 2x \le 0$

$D: -1 \le x \le 0$

ii. $x = \sin^{-1}(2y+1)$

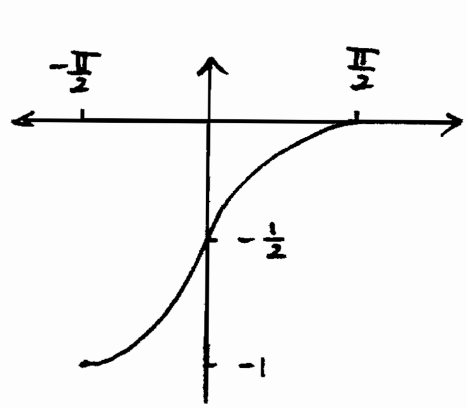
$\sin x = 2y+1$

$\therefore\ y = \frac{1}{2}(\sin x - 1)$

iii. $D: -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$R: -1 \le y \le 0$

iv.



Question 3

(a) $y = x\ln x$

i. $\frac{dy}{dx} = x\cdot\frac{1}{x} + \ln x\cdot 1$

$= 1+\ln x$

Stat pts: $\frac{dy}{dx} = 0$

$0 = 1+\ln x$

$\ln x = -1$

$x = \frac{1}{e},\ y = -\frac{1}{e}$

$\frac{d^2y}{dx^2} = \frac{1}{x}$

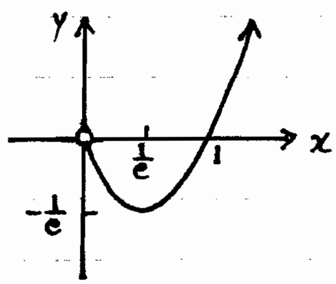
ii. $\frac{d^2y}{dx^2} = \frac{1}{x} > 0$ for all $x > 0$

$\therefore$ the curve is always concave up

iii. $\lim_{x \to 0} x \ln x = \underline{\underline{0}}$

iv. $\underline{\underline{(1, 0)}}$

v.



(b) $\int \frac{dx}{\sqrt{16 - 25x^2}} = \int \frac{dx}{\sqrt{25\left(\frac{16}{25} - x^2\right)}}$

$= \frac{1}{5} \int \frac{dx}{\sqrt{\frac{16}{25} - x^2}}$

$= \frac{1}{5} \sin^{-1} \frac{x}{4/5} + c$

$= \underline{\underline{\frac{1}{5} \sin^{-1} \frac{5x}{4} + c}}$

Question 4

(a) i. $P = 3000 + Ae^{kt}$

$\frac{dP}{dt} = kAe^{kt}$

$= k(P - 3000)$

since $P = 3000 + Ae^{kt}$

ii. when $t = 0$, $P = 4000$

$4000 = 3000 + Ae^{0}$

$\therefore \underline{\underline{A = 1000}}$

when $t = 1$, $P = 10\,000$

$10\,000 = 3000 + 1000e^{k}$

$7000 = 1000e^{k}$

$e^{k} = 7$

$\therefore \underline{\underline{k = \ln 7}}$

iii. when $t = 2.5$, $P = ?$

$P = 3000 + 1000e^{\ln 7 \times 2.5}$

$= 132\,641.8..$

$= \underline{\underline{132\,600}}$

iv. $P = 500\,000$, $t = ?$

$500\,000 = 3000 + 1000e^{(\ln 7)t}$

$497\,000 = 1000e^{(\ln 7)t}$

$497 = e^{(\ln 7)t}$

$t = \frac{\ln 497}{\ln 7}$

$= 3.19$ months

$\approx \underline{\underline{13 \text{ weeks}}}$

(b) $V = \frac{4}{3}\pi r^3$

$S = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$= 4\pi r^2 \frac{dr}{dt}$

$50 = 4\pi \times 20^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{5}{160\pi}$

$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

$= 8\pi r \times \frac{dr}{dt}$

$= 8\pi \times 20 \times \frac{5}{\ldots}$

Question 5

a) i. $\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2\theta + 1 \, d\theta$

$= \frac{1}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\left(\frac{1}{2}\sin 2.\frac{\pi}{4} + \frac{\pi}{4}\right) - \left(\frac{1}{2}\sin 0 + 0\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right]$

$= \frac{\pi + 2}{8}$

ii. $\int_0^{\sqrt{2}} \sqrt{4 - x^2} \, dx$

$x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta \, d\theta$

$x = \sqrt{2}$: $\sqrt{2} = 2\sin\theta$

$\frac{1}{\sqrt{2}} = \sin\theta$

$\theta = \frac{\pi}{4}$

$x = 0$: $0 = 2\sin\theta$

$\theta = 0$

$= \int_0^{\frac{\pi}{4}} \sqrt{4 - 4\sin^2\theta} \times 2\cos\theta \, d\theta$

$4(1 - \sin^2\theta)$

$= \int_0^{\frac{\pi}{4}} \sqrt{4\cos^2\theta} \times 2\cos\theta \, d\theta$

$= 4\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta$

$= 4 \times \frac{\pi + 2}{8}$

$= \frac{\pi + 2}{2}$

(b) $\int_1^3 \frac{1}{(1+x)\sqrt{x}} \, dx$

$x = u^2$

$\frac{dx}{du} = 2u$

$dx = 2u \, du$

$x = 3$: $3 = u^2$

$u = \sqrt{3}$

$x = 1$: $1 = u^2$

$u = 1$

$= \int_1^{\sqrt{3}} \frac{1}{(1+u^2)\,u} \times 2u \, du$

$= 2\int_1^{\sqrt{3}} \frac{1}{1+u^2} \, du$

$= 2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$

$= 2\left[\tan^{-1}\sqrt{3} - \tan^{-1} 1\right]$

$= 2\left[\frac{\pi}{3} - \frac{\pi}{4}\right]$

$= \frac{\pi}{6}$

Question 6

i. $x^4 + x^2 - 1 = 0$

let $x^2 = a$

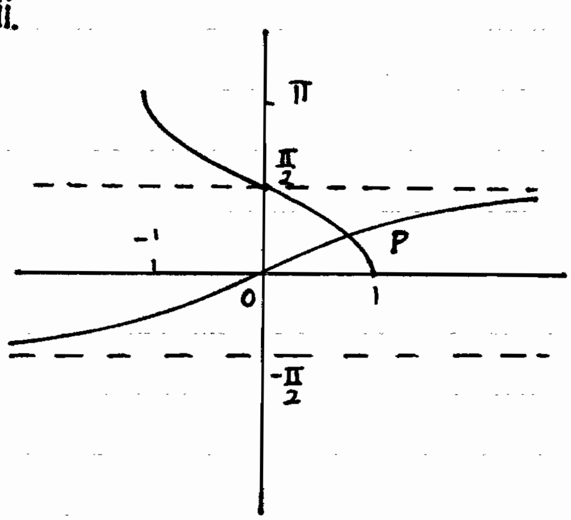
$a^2 + a - 1 = 0$

$a = \dfrac{-1 \pm \sqrt{1^2 - 4.1.-1}}{2}$

$x^2 = \dfrac{-1 \pm \sqrt{5}}{2}$

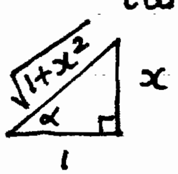
$\therefore \; x = \pm\ 0 \cdot 79$

ii.



iii. let $\tan^{-1} x = \alpha$

$\tan \alpha = x$



$\therefore \; \cos \alpha = \dfrac{1}{\sqrt{1+x^2}}$

iv. If $\tan^{-1} x = \cos^{-1} x$

$\cos^{-1} x = \alpha$

$\cos \alpha = x$

$\dfrac{1}{\sqrt{1+x^2}} = x$

$\dfrac{1}{1+x^2} = x^2$

v. $P(0 \cdot 79,\ 0 \cdot 67)$

vi. $y = \tan^{-1} x$

$x = \tan y$

$y = \cos^{-1} x$

$x = \cos y$

$A = \displaystyle\int_{0 \cdot 67}^{\frac{\pi}{2}} \cos y \, dy + \int_0^{0 \cdot 67} \tan y \, dy$

$= \displaystyle\int_{0 \cdot 67}^{\frac{\pi}{2}} \cos y \, dy + \int_0^{0 \cdot 67} \frac{\sin y}{\cos y} \, dy$

$= \left[\sin y\right]_{0 \cdot 67}^{\frac{\pi}{2}} + \left[-\ln(\cos y)\right]_0^{0 \cdot 67}$

$= \left[\sin \frac{\pi}{2} - \sin 0 \cdot 67\right] + \left[(-\ln(\cos 0 \cdot 67)) - (-\ln(\cos 0))\right]$

$= 0 \cdot 62$ (2 dp)