Name:______________________________

Class:______________________________

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12

HSC ASSESSMENT TASK 3

JUNE 2008

MATHEMATICS Extension 1

Time Allowed: 70 minutes

Instructions:

- Write your name and class at the top of each page
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start <u>each</u> question on a <u>new page.</u>
- Standard integrals can be found on the last page.

Question 1	Question 2	Question 3	Question 4	Question 5	Total
/12	/12	/12	/12	/12	/60

QUESTION1 - 12 Marks

a) Find i) $\int \frac{x}{9+x^2}\, dx$ **2**

ii) $\int \frac{1}{9+x^2}\, dx$ **2**

b) Solve for x

$\log_2 x = \log_2 10 - \log_2(x-3)$ **3**

c) Differentiate i) 5^x **1**

ii) $x^2 \sin^{-1} 2x$ **2**

d) Find the exact value of $\tan\left(\cos^{-1}\left(\frac{-3}{4}\right)\right)$ **2**

QUESTION 2 (Start a new page) - 12 Marks

a) Solve $\cos^2\theta - \sin^2\theta = 0.1$ for $0 \leq \theta \leq \pi$ **3**
(answer(s) in radians correct to 2 decimal places)

b) Find the general solution for $\sin\theta = \frac{1}{\sqrt{2}}$ **2**

c) i) Write $x^2 + 6x + 10$ in the form $(x+a)^2 + b$ **1**

ii) Hence find $\int \frac{dx}{x^2 + 6x + 10}$ **2**

d) i) Sketch $y = \sin^{-1} x$ **1**

ii) Find the exact area bounded by

$y = \sin^{-1} x$, the x axis and the line $x = 1$ **3**

QUESTION 3 **(Start a new page) - 12 Marks**

a) i) Find $\frac{d}{dx}\sqrt{1-x^2}$ **1**

ii) Using part i) show that

$\int_0^1 \frac{1+x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} + 1$ **3**

b) i) If $f(x) = (x-1)^2$ for $x \leq 1$, find $f^{-1}(x)$ and state its domain and range **3**

ii) Find any points(s) of intersection of $y = f(x)$ and $y = f^{-1}(x)$ **2**

c) i) If $\tan^{-1} x = \alpha$ and $\tan^{-1} y = \beta$ prove that

$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ **2**

ii) Hence evaluate $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$ (in exact form) **1**

QUESTION 4 **(Start a new page) - 12 Marks**

a) Find $\int \frac{dx}{\sqrt{x}(1+x)}$ using the substitution $u = \sqrt{x}$ or otherwise **3**

b) A cylindrical solid of height 10cm is being turned on a cutting machine so that the radius is being reduced by 0.3cm/min.

Find at what rate the surface area is decreasing, when the radius is 5cm (in exact form)

(surface area = $2\pi r^2 + 2\pi rh$) 3

c) The rate of cooling of an object is proportional to the excess of the object's temperature above the surrounding temperature, $\frac{dT}{dt} = k\ (T - T_0)$

T is the object's temperature
T_0 is the surrounding temperature.

A pot of hot water cools from 90°C *to* 85°C in 1 minute at a room temperature of 30°C.

i) Show that $T = T_0 + Ae^{kt}$ satisfies the above equation **1**

ii) Find the exact values of A and k. **2**

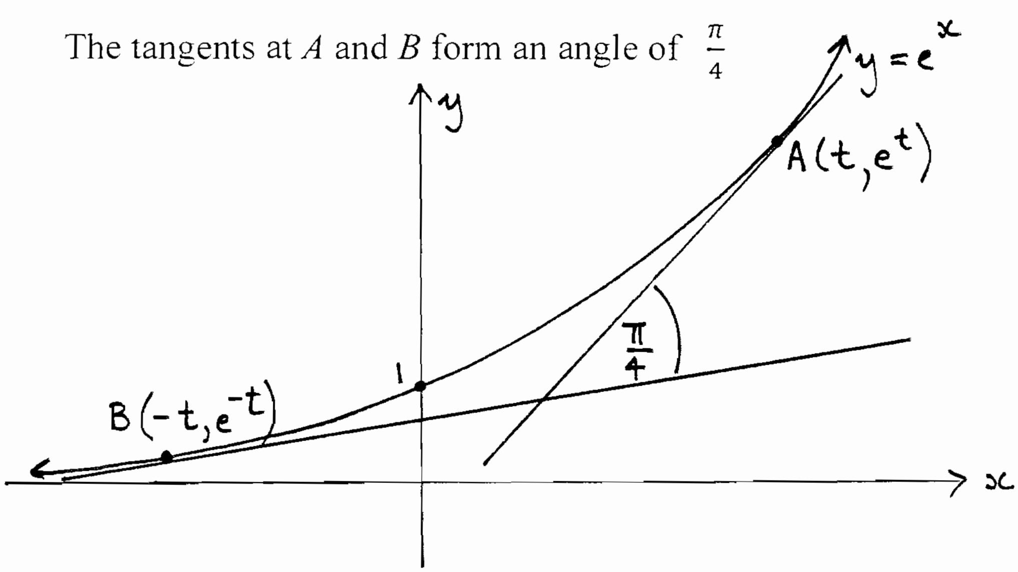
iii) How long would it take to cool to 60°C? (nearest second) **2**

iv) What would be the temperature after 4 minutes? (2 dec. places) **1**

QUESTION 5 (Start a new page) 12 Marks

a) A (t, e^t) and B $(-t, e^{-t})$ are points on the curve $y = e^x$, where $t > 0$.

The tangents at A and B form an angle of $\frac{\pi}{4}$



i) Prove that $e^t - e^{-t} = 2$ **2**

ii) Solve this equation to prove $t = ln\left(\sqrt{2} + 1\right)$ **2**

b) Find $\int \sin^2 3x \ dx$ 2

c) P is the point of intersection of the graphs $y \tan x$ and $y = A\sin x$ where $A > 1$. The x co – ordindate of P is α, and α lies between 0 and $\frac{\pi}{2}$

i) Sketch $y = \tan x$ *and* $y = A \sin x$ on the same axes for $0 \leq x \leq \frac{\pi}{2}$ Label the point P **2**

ii) Prove $\cos \alpha = \frac{1}{A}$ at P **1**

iii) If 0 is the origin, prove that the area enclosed by the arcs OP, on both graph is (A – 1 – 1nA) unit ^{2} **3**

(End of Paper)

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

HSC Assec Task 3 June 2008

Question 1

a) i) $\int \frac{x}{9+x^2}\,dx = \frac{1}{2}\int \frac{2x}{9+x^2}\,dx$

$= \frac{1}{2}\ln(9+x^2)+c$

ii) $\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3}+c$

b) $\log_2 x = \log_2 10 - \log_2(x-3)$

$\log_2 x = \log_2\left(\frac{10}{x-3}\right)$

$x = \frac{10}{x-3}$

$x^2 - 3x = 10$

$x^2 - 3x - 10 = 0$

$(x-5)(x+2)=0 \quad \therefore x = 5$ only as $x>0$

c) i) $\frac{d}{dx}(5^x) = \ln 5 \cdot 5^x$

ii) $u = x^2 \qquad v = \sin^{-1}2x$

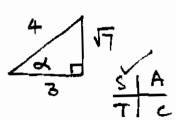
$u' = 2x \qquad v' = \frac{2}{\sqrt{1-4x^2}}$

$\therefore \frac{d}{dx}(x^2 \cdot \sin^{-1}2x) = 2x\sin^{-1}2x + \frac{2x^2}{\sqrt{1-4x^2}}$

d) $\tan\left(\cos^{-1}\left(-\frac{3}{4}\right)\right) = \tan\left(\pi - \cos^{-1}\frac{3}{4}\right)$

Let $\alpha = \cos^{-1}\frac{3}{4}$

$\cos\alpha = \frac{3}{4}$



$\therefore \tan(\pi-\alpha) = -\tan\alpha$

$= -\frac{\sqrt{7}}{3}$

Question 2

a) $\cos^2\theta - \sin^2\theta = 0.1$

$\cos 2\theta = 0.1$

$2\theta = 1.4706,\ 4.8126$

$\theta = 0.74,\ 2.41$

S	A ✓
T	C ✓

b) $\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\theta = n\pi + (-1)^n \cdot \frac{\pi}{4}$

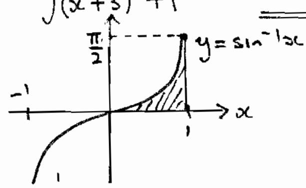
where n is an integer

c) i) $x^2 + 6x + 10 = (x^2+6x+9)+1$

$= (x+3)^2+1$

ii) $\int \frac{dx}{(x+3)^2+1} = \tan^{-1}(x+3)^2 + c$

d)



$A = \int_0^{\pi/2} \sin^{-1}x\,dx$

$= \frac{\pi}{2} - \int_0^{\pi/2} \sin y\,dy$

$= \frac{\pi}{2} - \left[-\cos y\right]_0^{\pi/2}$

$= \frac{\pi}{2} + [0-1]$

$= \left(\frac{\pi}{2}-1\right)$ unit^2

Question 3

a) i) $\frac{d}{dx}(1-x^2)^{1/2} = \frac{1}{2}\times -2x(1-x^2)^{-1/2}$

$= \frac{-x}{\sqrt{1-x^2}}$

ii) $\int_0^1 \frac{1+x}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\,dx$

$= \left[\sin^{-1}x - \sqrt{1-x^2}\right]_0^1$

$= \frac{\pi}{2} - (0-1)$

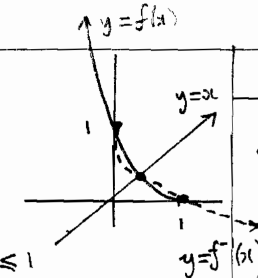
$= \frac{\pi}{2}+1$

b) i) $f(x) = (x-1)^2$

$x = (y-1)^2$

$\pm\sqrt{x} + 1 = y$

$f^{-1}(x) = -\sqrt{x}+1$



D: $x \geq 0$ R: $y \leq 1$

ii) pt. int $x = (x-1)^2$

$x^2 - 3x + 1 = 0$

$x = \frac{3 \pm \sqrt{9-4\times1\times1}}{2}$

$x = \frac{3\pm\sqrt{5}}{2}$ from above D and R

at $x = \frac{3-\sqrt{5}}{2}$

c) i) $\tan^{-1}x = \alpha \qquad \tan^{-1}y = \beta$

$\therefore x = \tan\alpha \qquad y = \tan\beta$

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha.\tan\beta}$

$\tan(\alpha+\beta) = \frac{x+y}{1-xy}$

$\alpha+\beta = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

ii) $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$

$= \tan^{-1}(1)$

$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

Question 4

a) $u = \sqrt{x} = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} =$

$du = \frac{1}{2\sqrt{x}}\,dx$

$\therefore dx = du \cdot 2 \cdot \sqrt{x}$

$\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{du \cdot 2\sqrt{x}}{\sqrt{x}(1+u^2)}$

$= 2\int \frac{du}{1+u^2}$

$= 2\tan^{-1}u +$

$= 2\tan^{-1}\sqrt{x} +$

b) $A = 2\pi R^2 + 2\pi R.10$

$\frac{dR}{dt} = -.3$ cm/min

$\frac{dA}{dR} = 4\pi R + 20\pi$

$\frac{dA}{dt} = \frac{dA}{dR} \cdot \frac{dR}{dt}$

$= (4\pi R + 20\pi) \times -.3$

sub $R=$

$= 40\pi \times -.3$

$\frac{dA}{dt} = -12\pi$ cm^2/min

c) i) $T = T_0 + Ae^{kt}$

$\frac{dT}{dt} = k\,Ae^{-kt}$

since $T - T_0 = Ae^{kt}$

$\therefore \frac{dT}{dt} = k(T-T_0)$

ii) $T = 30 + Ae^{kt}$ sub

$90 = 30 + A$

$\therefore A = 60$

$A = 60 \quad T = 85 \quad t = 1$

$5 = 30 + 60e^{k}$

$\frac{55}{60} = e^{k}$

$\ln\left(\frac{11}{12}\right) = k$

$= 60^\circ$ find t

$= 30 + 60e^{\ln(\frac{11}{12})t}$

$0 = 60e^{\ln(\frac{11}{12})t}$

$\frac{1}{2}) = \ln\left(\frac{11}{12}\right)t$

$t = \frac{\ln(1/2)}{\ln(11/12)}$

$t = 7.97\text{min}$

$t = 7\text{min } 58\text{sec}$

$= 4$ find T

$T = 30 + 60e^{\ln(\frac{11}{12}).4}$

$T = 72.36^\circ C$

tion 5

$y = e^x \qquad \frac{dy}{dx} = e^x$

at A

$-t$

at B

$\frac{\pi}{4} = \left|\frac{e^t - e^{-t}}{1 + e^t . e^{-t}}\right|$

$1 = \frac{e^t - e^{-t}}{1+1}$

$e^t - e^{-t} = 2$

ii) Let $u = e^t$

$u - u^{-1} = 2$

$u - \frac{1}{u} = 2$

$u^2 - 1 = 2u$

$u^2 - 2u - 1 = 0$

$u = \frac{2 \pm \sqrt{4 - 4\times1\times-1}}{2}$

$u = \frac{2 \pm \sqrt{8}}{2} = \frac{2(1 \pm \sqrt{2})}{2}$

$\therefore u = 1 \pm \sqrt{2}$

$\therefore$ $e^t = 1 + \sqrt{2}$ or $e^t = 1 - \sqrt{2}$

$\ln(e^t) = \ln(1+\sqrt{2})$ $\qquad e^t > 0$ $\therefore$ no solution

$t = \ln(1+\sqrt{2})$ only

b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= (1 - \sin^2\theta) - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\therefore \int \sin^2 3x \, dx$

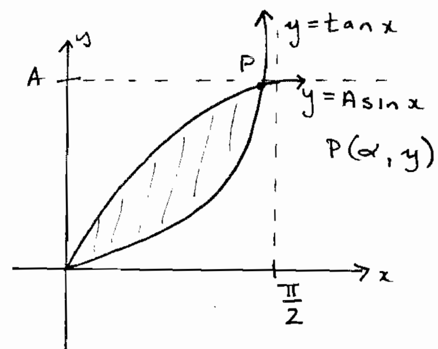
$= \frac{1}{2}\int(1 - \cos 6x)\, dx$

$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right] + c$

$= \frac{x}{2} - \frac{1}{12}\sin 6x + c$

c)

i)



ii) P pt of intersection

$\therefore \tan\alpha = A\sin\alpha$

$\frac{\sin\alpha}{\cos\alpha} = A\sin\alpha$

$\therefore \frac{1}{\cos\alpha} = A \Rightarrow \cos\alpha = \frac{1}{A}$

iii) $a = \int_0^\alpha (A\sin x - \tan x)\, dx$

$= \int_0^\alpha \left(A\sin x - \frac{\sin x}{\cos x}\right) dx$

$= \left[-A\cos x + \ln(\cos x)\right]_0^\alpha$

$= (-A\cos\alpha + \ln(\cos\alpha)) - (-A)$

$= -A.\frac{1}{A} + \ln\left(\frac{1}{A}\right) + A$

Area $= -1 + \ln A^{-1} + A = (A - 1 - \ln A)\,\text{unit}^2$