Name:_____

Class:_____

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12

HSC ASSESSMENT TASK 3

JUNE 2009

MATHEMATICS Extension 1

Time Allowed: 70 minutes

Instructions:

- Write your name and class at the top of each page.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start **each** question on a **new page**.
- Standard integrals can be found on the last page.

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total
/10	/11	/10	/10	/11	/11	/63

Question 1 **Marks**

a) Solve $log_e x + log_e(x-3) = log_e 4$ 3

b) Evaluate and leave in exact form

(i) $cos^{-1}(\frac{-\sqrt{3}}{2})$ 1

(ii) $\sin(2\ cos^{-1}\frac{4}{5})$ 2

c) Find (i) $\int \frac{2}{9+x^2}\ dx$ 2

(ii) $\int \frac{dx}{\sqrt{1-9x^2}}$ 2

Question 2

a) Consider the function $f(x) = 8x - x^2$

(i) Sketch $y = f(x)$ clearly showing the x and y intercepts and the vertex. 2

(ii) State the largest domain containing $x = 8$ for which $f(x)$ has an inverse function, $f^{-1}(x)$ 1

(iii) State the domain of $f^{-1}(x)$ 1

(iv) Find the equation of $f^{-1}(x)$ 2

(v) For what value(s) of x does $f(x) = f^{-1}(x)$? 2

(vi) Evaluate $f^{-1}(f(-1))$ 1

b) Sketch the function $y = sin^{-1}x$ showing clearly the domain and range. 2

Question 3 **Marks**

Differentiate

(i) $y = e^{3x} log_e 3x$ 2

(ii) $y = log_e(\frac{x-2}{x})$ 2

(iii) $y = log_3 x$ 2

(iv) $y = log_e(cosx)$ (give answer in simplified form) 2

(v) $y = \tan(log_e x)$ 2

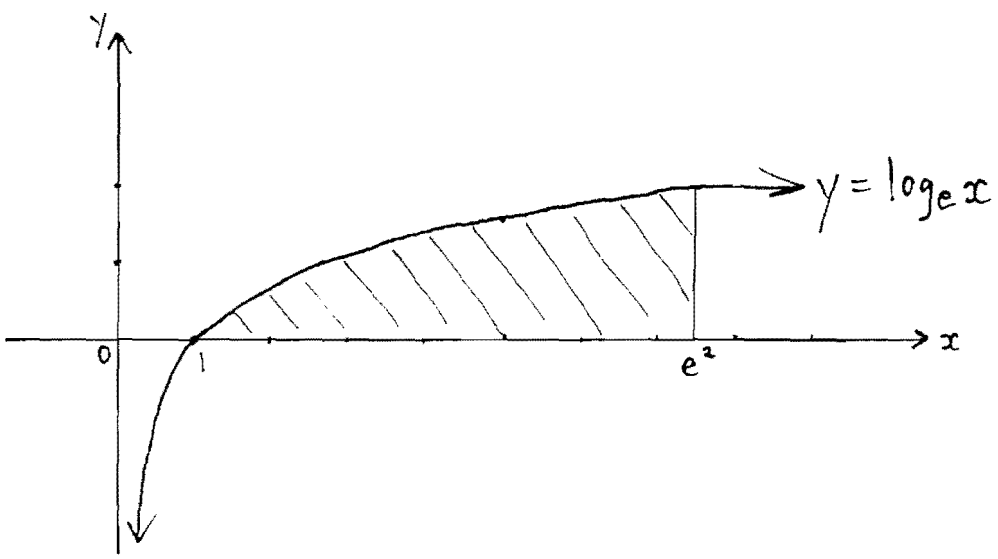
Question 4

a) (i) Express $sin^2 x$ in terms of $cos2x$ 1

(ii) Use this result or otherwise to evaluate $\int_0^{\frac{\pi}{4}} sin^2 2x dx$ 3

b) Differentiate $sin^2 3x$ and hence evaluate $\int_0^{\frac{\pi}{6}} sin3xcos3x dx$ 3

c) 3



Find the shaded area and leave it in exact form.

Question 5 **Marks**

a) Find the general solution for $sin\theta = \frac{1}{\sqrt{2}}$ 2

b) Differentiate (i) $y = tan^{-1}3x$ 2

(ii) $y = sin^{-1}(cosx)$ 2

c) Sketch $y = 1 - 3sin2x$ between $0 \leq x \leq 2\pi$ 3

d) Sketch $y = cos^{-1}(\cos x)$ 2

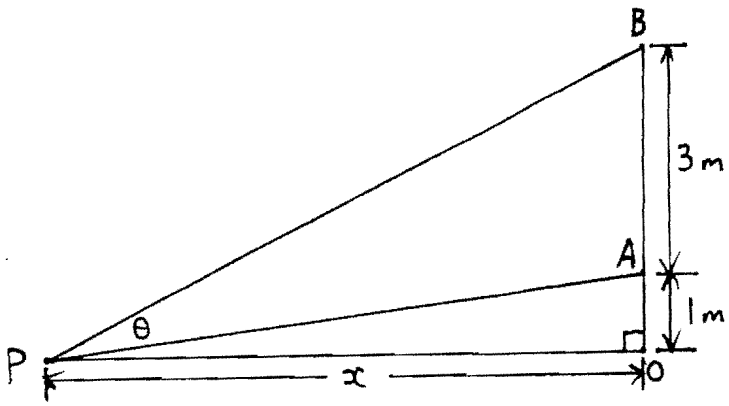
Question 6

a) A spherical balloon is being inflated so that the surface area is increasing at the rate of $0.3cm^2s^{-1}$. When the balloon's radius is 4cm, find the rate of increase. $\left[v = \frac{4}{3}\pi r^3, SA = 4\pi r^2\right]$

i) In the radius (correct to 3 decimal places) 2

ii) In the volume (correct to one decimal place) 2

b) In the diagram, a vertical pole AB, 3 metres high, is placed on top of a 1 metre high support. The pole subtends an angle of θ radians at the point P, which is x metres from the base O of the support.



(i) Show that $\theta = tan^{-1}\frac{4}{x} - tan^{-1}\frac{1}{x}$ 2

(ii) Show that θ has a stationary point when $x = 2$. 3
Assume it is a maximum.

(iii) Deduce that the maximum angle subtended at P is $\theta = tan^{-1}\frac{3}{4}$. 2

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2009 Ext. 1 Task 3 Solutions

Question 1

a) $\log_e x + \log_e (x-3) = \log_e 4$

$\log_e x(x-3) = \log_e 4$

$x^2 - 3x = 4$

$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

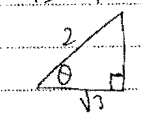
$x = -1$ or 4

but x must be > 0

$\therefore x = 4$

b) (i) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

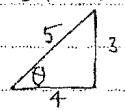
Range of $\cos^{-1}$ is $0 \leq \theta \leq \pi$ $\therefore \theta$ is in 2nd quadrant



$\theta = \pi - \frac{\pi}{6}$

$\theta = \frac{5\pi}{6}$

(ii) $\sin\left(2\cos^{-1}\frac{4}{5}\right)$



Let $\cos^{-1}\frac{4}{5} = \theta$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta$

$= 2 \times \frac{3}{5} \times \frac{4}{5}$

$= \frac{24}{25}$

c) (i) $\int \frac{2}{9+x^2}\,dx$

$\frac{2}{3}\int \frac{3}{9+x^2}\,dx$

$= \frac{2}{3}\tan^{-1}\frac{x}{3} + c$

(ii) $\int \frac{dx}{\sqrt{1-9x^2}}$

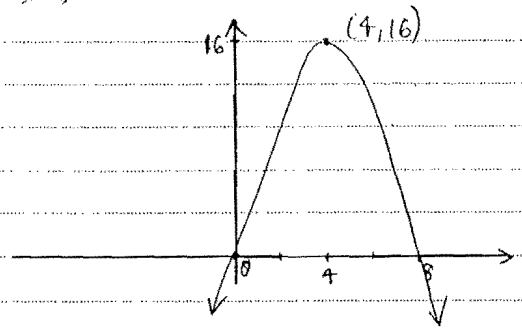
$\frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}}$

$\frac{1}{3}\sin^{-1}\frac{x}{\frac{1}{3}}$

$\frac{1}{3}\sin^{-1} 3x + c$

Question 2

a) (i) $f(x) = 8x - x^2$



(ii) $x \geq 4$

(iii) $x \leq 16$

(iv) $y = 8x - x^2$

$x = 8y - y^2$

$y^2 - 8y + 16 = -x + 16$

$(y-4)^2 = -x + 16$

$y - 4 = \pm\sqrt{16-x}$

$y = 4 + \sqrt{16-x}$

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(v) They intersect on $y = x$

$\therefore$ Solve $y = x$ and $y = 8x - x^2$

$x = 8x - x^2$

$x^2 = 7x$

$x(x-7) = 0$

$x = 7 \Rightarrow (7,7)$

(vi) $f^{-1}(f(-1))$

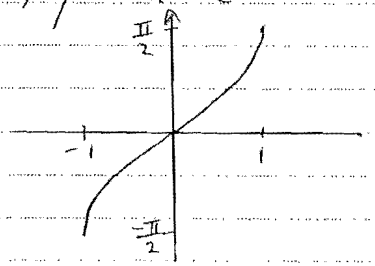
$= f^{-1}(-9)$

$= 4 + \sqrt{16 - -9}$

$= 4 + 5$

$= 9$

b) $y = \sin^{-1} x$



Question 3

(i) $y = e^{3x}\log_e 3x$

$y' = e^{3x} \times \frac{3}{3x} + 3e^{3x}\log_e 3x$

$= \frac{e^{3x}}{x} + 3e^{3x}\log_e 3x$

(ii) $y = \log_e\left(\frac{x-2}{x}\right)$

$= \log_e(x-2) - \log_e x$

$y' = \frac{1}{x-2} - \frac{1}{x}$

(iii) $y = \log_3 x$

$y = \frac{\log_e x}{\log_e 3}$

$y' = \frac{1}{x\log_e 3}$

(iv) $y = \log_e(\cos x)$

$y' = \frac{-\sin x}{\cos x}$

$y' = -\tan x$

(v) $y = \tan(\log_e x)$

$y' = \sec^2(\log_e x) \times \frac{1}{x}$

$y' = \frac{\sec^2(\log_e x)}{x}$

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Question 4

a)(i) $\sin^2 x = \frac{1-\cos 2x}{2}$

(ii) $\int_0^{\frac{\pi}{4}} \sin^2 2x \, dx$

$\int_0^{\frac{\pi}{4}} \frac{1-\cos 4x}{2} \, dx$

$\frac{1}{2}\left[x - \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{4}}$

$\frac{1}{2}\left[\frac{\pi}{4} - 0 - (0)\right]$

$= \frac{\pi}{8}$

b) $\frac{d}{dx}(\sin^2 3x) = 2\sin 3x \cos 3x \times 3$

$= 6\sin 3x \cos 3x$

$\therefore \frac{1}{6}\frac{d}{dx}(\sin^2 3x) = \sin 3x \cos 3x$

$\therefore \frac{1}{6}\int_0^{\frac{\pi}{6}} \frac{d}{dx}(\sin^2 3x)\,dx$

$= \int_0^{\frac{\pi}{6}} \sin 3x \cos 3x \, dx$

$LHS = \frac{1}{6}\left[\sin^2 3x\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{6}\left(\sin\frac{\pi}{2}\right)^2$

$= \frac{1}{6}$

c) Area shaded = Rectangle − Area to y axis

$= e^2 \times 2 - \int_0^2 x \, dy$

$= 2e^2 - \int_0^2 e^y \, dy$

$= 2e^2 - \left[e^y\right]_0^2$

$= 2e^2 - (e^2 - e^0)$

$= e^2 + 1$

Question 5

a) $\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = n\pi + (-1)^n \sin^{-1}\frac{1}{\sqrt{2}}$

$\theta = n\pi + (-1)^n \times \frac{\pi}{4}$

b)(i) $y = \tan^{-1} 3x$

$y = \tan^{-1}\frac{x}{\frac{1}{3}}$

$y' = \frac{\frac{1}{3}}{x^2 + (\frac{1}{3})^2}$

$= \frac{3}{9x^2+1}$

(ii) $y = \sin^{-1}(\cos x)$

$y' = \frac{-\sin x}{\sqrt{1-\cos^2 x}}$

$y' = -1$

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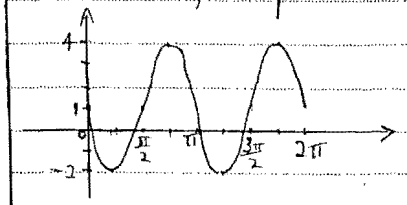
c) $y = 1 - 3\sin 2x$

Period π, Amplitude 3

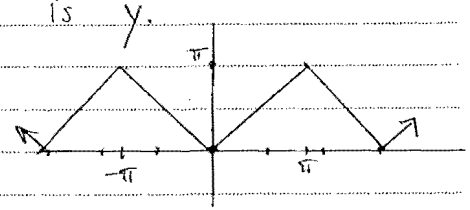


d) $y = \cos^{-1}(\cos x)$

Domain: All real x

Range: $0 \le y \le \pi$

Since $\cos x$ is even, so is y.



Question 6

a) $\frac{dA}{dt} = 0.3$ $A = 4\pi r^2$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$0.3 = 8\pi r \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{0.3}{8\times\pi\times 4}$

$\frac{dr}{dt} = 0.003\,cm/s$

(ii) $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$= 4\pi r^2 \times 0.003$

$= 4\times\pi\times 4^2 \times 0.003$

$= 0.6\,cm^3/s$

b)(i) $\tan\angle BPO = \frac{4}{x} \Rightarrow \angle BPO = \tan^{-1}\frac{4}{x}$ $\theta = \angle BPO - \angle APO$

$\tan\angle APO = \frac{1}{x} \Rightarrow \angle APO = \tan^{-1}\frac{1}{x}$ $\therefore \theta = \tan^{-1}\frac{4}{x} - \tan^{-1}$

(ii) $\frac{d\theta}{dx} = \frac{\frac{-4}{x^2}}{1+(\frac{4}{x})^2} - \frac{\frac{-1}{x^2}}{1+(\frac{1}{x})^2}$

$= \frac{-4}{x^2+16} + \frac{1}{x^2+1}$

$= \frac{-4x^2 - 4 + x^2 + 16}{(x^2+16)(x^2+1)}$

$\frac{d\theta}{dx} = 0$ if $-3x^2 + 12 = 0$

$x^2 = 4$

$x = 2, (x>0)$

(iii) Sub. $x = 2$ into

$\theta = \tan^{-1}\frac{4}{2} - \tan^{-1}\frac{1}{2}$

$\tan\theta = \tan\left[\tan^{-1}2 - \tan^{-}\right.$

$= \frac{\tan(\tan^{-1}2) - \tan(t}{1 + \tan(\tan^{-1}2)\tan(t}$

$= \frac{2 - \frac{1}{2}}{1 + 2\times\frac{1}{2}}$

$= \frac{1\frac{1}{2}}{2}$

$\therefore \theta = \tan^{-1}\frac{3}{4}$