Name: File

Teacher:

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12

HSC ASSESSMENT TASK 3

JUNE 2011

MATHEMATICS Extension 1

Time Allowed: 70 minutes

Instructions:

- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start **each** question on a **new page.**
- Standard integrals can be found on the last page.

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total
/10	/10	/10	/10	/10	/10	/60

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 (10 marks) **Marks**

a) Differentiate $e^{\sin x}$ 1

b) Find $\frac{d}{dx}(3^x)$ 1

c) Find the <u>exact</u> value of $\tan\left(\sin^{-1}\frac{1}{5}\right)$ 1

d) Solve $\sin 2\theta = \cos\theta$ for $0 \leq \theta \leq 2\pi$ 3

e) Find i) $\int \frac{1}{\sqrt{9-x^2}}\,dx$ 1

ii) $\int \sin\left(\frac{\pi}{4} - x\right) dx$ 2

iii) $\int 3xe^{x^2}\,dx$ 1

Question 2 (10 marks) Start a new page

a) If $\int_0^2 \frac{dx}{x^2+4} = k$ find k 2

b) i) Differentiate $f(x) = \tan^{-1}x + \tan^{-1}\frac{1}{x}$ if $x \neq 0$ 2

ii) Hence evaluate $\int_1^2 f(x)dx$ 1

c) i) Find $\frac{d}{dx}(x\ln x)$ 2

ii) Hence show that $\int_e^{e^2} \frac{1+\ln x}{x\ln x}\,dx = 1 + \ln 2$ 3

Question 3 (10 marks) Start a new page **Marks**

a) Find $\int_0^{\frac{\pi}{2}} \cos^2 4x \, dx$ 3

b) Sketch $y = 2sin^{-1}(x-1)$ and state its domain and range. 3

c) i) Write $y = cosx - \sqrt{3}\,sinx$ in the form $y = Acos(x+B)$, where B is acute. 2

ii) Hence, sketch $y = cosx - \sqrt{3}\,sinx$ for $0 \le x \le 2\pi$. Find and label clearly, the points where the curve cuts the x and y axes. 2

Question 4 (10 marks) Start a new page

a) An area of 1 square unit is bounded by the curve $y = \frac{1}{x}$, the x axis and the lines $x = e$ and $x = k$. Find k if $k > e$. 2

b) i) Show that $f(x) = \frac{1}{1+x^2}$ is an even function and then, without the use of calculus, make a neat sketch of $f(x) = \frac{1}{1+x^2}$. 2

ii) What is the largest domain, containing $x = -1$, for which $f(x)$ has an inverse function $f^{-1}(x)$? 1

iii) Find $f^{-1}(x)$ 2

iv) Sketch $y = f^{-1}(x)$ 1

c) Find the general solution for $\sin x = \frac{1}{2}$ 2

Question 5 **(10 marks)** **Start a new page** **Marks**

a) i) Show that $x^2 + 6x + 10 = (x+3)^2 + 1$ 1

ii) Hence find $\int \frac{1}{x^2+6x+10}\, dx$ 1

b) i) Differentiate $x tan^{-1} x$ 1

ii) Hence find $\int tan^{-1} x \ \ dx$ 2

c) An object, removed from a freezer at $-5°C$, is placed in a room where the temperature is kept at a constant $15°C$. Thereafter, its temperature $T°C$, is changing so that after t minutes $\frac{dT}{dt} = k(15 - T)$, where k is a constant

i) show that $T = 15 - Ae^{-kt}$ satisfies this differential equation 1

ii) Find the value of A 1

iii) If initially, the temperature was increasing at $5°C$ per min, find the value of k. 1

iv) Find the temperature of the object, 5 mins after it was placed in the room (correct to 2 dec. pl.) 1

v) Find to, the nearest second , the time taken for the temperature of the object to rise to $0°C$. 1

Question 6 (10 marks) Start a new page

a) A square metal plate has sides $x\ cm$ and an area of $A cm^2$. It is expanding, so that the sides are increasing at 0.08 cm/min. Find the rate at which the area is increasing, when the sides are 7cm long. 3

b) i) Sketch $y = 1 - \tan x$, in the domain $0 \leq x \leq \frac{\pi}{4}$, without the use of calculus. 2

ii) Prove the area of the region enclosed by the above curve, the x axis and the y axis is $\frac{\pi - \ln 4}{4}$ square units. 2

iii) The region in part ii) is rotated around the x axis.
Find the volume of the solid formed. 3

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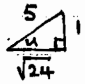
Question 1

a) $\frac{d}{dx}\left(e^{\sin x}\right) = \cos x . e^{\sin x}$

b) $\frac{d}{dx}\left(3^{x}\right) = \ln 3 . 3^{x}$

c) Let $u = \sin^{-1}\frac{1}{5}$

$\sin u = \frac{1}{5}$



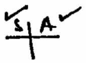
$\therefore \tan u = \tan\left(\sin^{-1}\frac{1}{5}\right) = \frac{1}{\sqrt{24}}$

d) $\sin 2\theta = \cos\theta$

$2\sin\theta . \cos\theta - \cos\theta = 0$

$\cos\theta\,(2\sin\theta - 1) = 0$

$\cos\theta = 0 \qquad \sin\theta = \frac{1}{2}$



$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

e) i) $\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\frac{x}{3} + c$

ii) $\int \sin\left(\frac{\pi}{4} - x\right)dx$

$= \cos\left(\frac{\pi}{4} - x\right) + c$

iii) $\int 3x . e^{x^2}\,dx = \frac{3x\,e^{x^2}}{2x} + c$

$= \frac{3e^{x^2}}{2} + c$

b i) $f(x) = \tan^{-1}x + \tan^{-1}(x^{-1})$

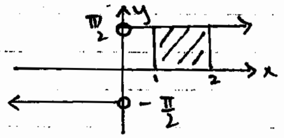
$f'(x) = \frac{1}{1+x^2} + \frac{-x^{-2}}{1+x^{-2}}$

$= \frac{1}{1+x^2} - \left[\frac{1}{x^2} \cdot \frac{x^2}{x^2+1}\right]$

$= \frac{1}{1+x^2} - \frac{1}{x^2+1}$

$f'(x) = 0 \quad \text{if} \quad x \neq 0$

ii)



$\therefore \int_1^2 f(x)\,dx = 1 \times \frac{\pi}{2}$

$= \frac{\pi}{2}$

c) i) $u = x \qquad v = \ln x$

$u' = 1 \qquad v' = \frac{1}{x}$

$\frac{d}{dx}(x\ln x) = \ln x + 1$

ii) $\int_e^{e^2} \frac{1+\ln x}{x\ln x}\,dx = \left[\ln(x\ln x)\right]_e^{e^2}$

$= \ln(e^2 . \ln e^2) - \ln(e \ln e)$

$= \ln(e^2 . 2\ln e) - 1$

$= \ln 2e^2 - 1$

$= \ln 2 + 2\ln e - 1$

$= \ln 2 + 1$

Question 2

a) $\int_0^2 \frac{dx}{x^2+4} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

$= \frac{1}{2}\left(\tan^{-1}1 - \tan^{-1}0\right)$

$k = \frac{\pi}{8}$

Question 3

a) since $\cos^2\theta = \frac{1}{2}\left[\cos 2\theta + 1\right]$

$\therefore \int_0^{\pi/2} \frac{1}{2}\left(\cos 8x + 1\right)dx$

$\frac{1}{2}\left[\frac{1}{8}\sin 8x + x\right]_0^{\pi/2} = \frac{1}{2}\left[\frac{1}{8}\sin 4\pi + \frac{\pi}{2}\right]$

$= \pi/4$

b) $y = 2\sin^{-1}(x-1)$

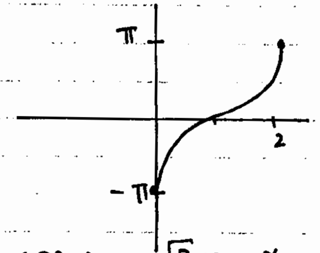
$\frac{y}{2} = \sin^{-1}(x-1)$

$-\frac{\pi}{2} \leq \frac{y}{2} \leq \frac{\pi}{2}$

$\therefore R: -\pi \leq y \leq \pi$

$-1 \leq x-1 \leq 1$

$D: 0 \leq x \leq 2$



c) $y = \cos x - \sqrt{3}\sin x$

$A = \sqrt{1+3} = 2$

i) $2\left[\cos x . \cos\beta - \sin x \sin\beta\right]$

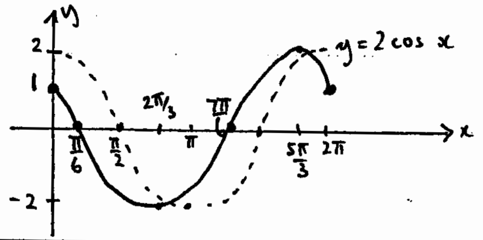
$2\left[\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right]$

$\cos B = \frac{1}{2} \qquad \sin B = \frac{\sqrt{3}}{2}$

$\therefore \beta = \frac{\pi}{3}$

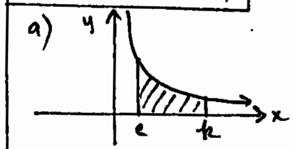
$\therefore y = \cos x - \sqrt{3}\sin x$ can be written $y = 2\cos\left(x + \frac{\pi}{3}\right)$

ii)



Question 4

a)



$1 = \int_e^k \frac{1}{x}\,dx$

$1 = \left[\ln x\right]_e^k$

$1 = \ln k - 1$

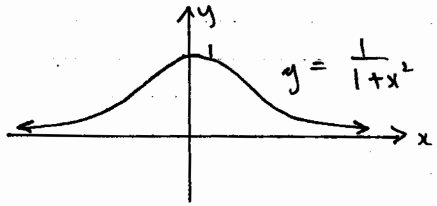
$\therefore \log_e k = 2$

$\therefore e^2 = k$

b) i) $f(x) = \frac{1}{1+x^2}$

$f(-x) = \frac{1}{1+(-x)^2} = \frac{1}{1+x^2}$

$\therefore f(x) = f(-x)$ even



ii) $D: x \leq 0$

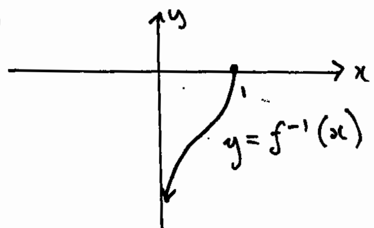
iii) $x = \frac{1}{1+y^2}$

$\frac{1}{x} = 1 + y^2$

$\frac{1}{x} - 1 = y^2$

$\therefore f^{-1}(x) = -\sqrt{\frac{1}{x} - 1}$

iv)



c) $\sin x = \frac{1}{2}$

$x = n\pi + (-1)^n \sin^{-1}\frac{1}{2}$

$= n\pi + (-1)^n \frac{\pi}{6}$

where n is an integer

Question 5

a) i) RHS $= (x+3)^2 + 1$

$= x^2 + 6x + 9 + 1$

$= x^2 + 6x + 10$

$=$ LHS

ii) $\int \frac{1}{x^2+6x+10}\,dx = \int \frac{1}{(x+3)^2+1}\,dx$

$= \tan^{-1}(x+3) + c$

b) i) $u = x \qquad v = \tan^{-1}x$

$u' = 1 \qquad v' = \frac{1}{1+x^2}$

$\frac{d}{dx}(x\tan^{-1}x) = \tan^{-1}x + \frac{x}{1+x^2}$

$\therefore \tan^{-1}x = \frac{d}{dx}(x.\tan^{-1}x) - \frac{x}{1+x^2}$

$\therefore \int \tan^{-1}x\,dx = \int\left(\frac{d}{dx}(x.\tan^{-1}x) - \frac{x}{1+x^2}\right)dx$

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$

c) i) $\frac{dT}{dt} = k(15 - T) \qquad t=0 \quad T=-5$

given $T = 15 - Ae^{-kt}$

$\frac{dT}{dt} = kAe^{-kt}$ *

since $15 - T = Ae^{-kt}$

$\therefore \frac{dT}{dt} = k(15-T)$

ii) $-5 = 15 - Ae^0$

$A = 20 \qquad \therefore T = 15 - 20e^{-kt}$

iii) $t = 0 \qquad \frac{dT}{dt} = 5$ sub into *

$5 = k.20e^0$

$\frac{5}{20} = k \qquad \therefore k = \frac{1}{4}$

iv) $t = 5$

$T = 15 - 20e^{-\frac{1}{4}.5}$

$T = 9.27°C$

v) $0 = 15 - 20e^{-\frac{1}{4}.t}$

$20e^{-t/4} = 15$

$e^{-t/4} = \frac{3}{4}$

$-t/4 = \ln\frac{3}{4}$

$t = 1.1507\ldots$ min

$\therefore t = 69$ seconds

Question 6

a) side $x \qquad A = x^2$

$\frac{dx}{dt} = 0.08 \qquad \frac{dA}{dx} = 2x$

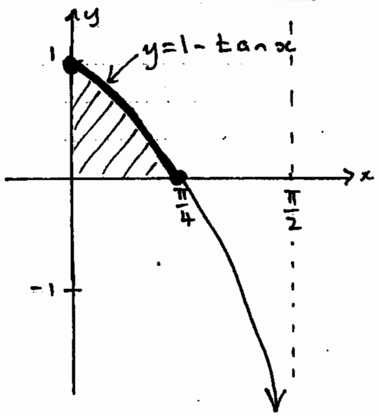
$\therefore \frac{dA}{dt} = \frac{dA}{dx}.\frac{dx}{dt}$

$= 2x \times 0.08$

sub $x = 7$

$\therefore \frac{dA}{dt} = 1.12$ cm/min

b) i)



ii) $A_x = \int_0^{\frac{\pi}{4}} (1 - \tan x)\,dx$

$= \int_0^{\pi/4} 1 - \frac{\sin x}{\cos x}\,dx$

$= \left[x + \ln\cos x\right]_0^{\pi/4}$

$= \left(\frac{\pi}{4} + \ln\cos\frac{\pi}{4}\right) - (0 + \ln\cos 0)$

$= \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}$

$= \frac{\pi}{4} + \ln 2^{-1/2}$

$= \pi/4 - \frac{1}{2}\ln 2$

$= \frac{\pi - 2\ln 2}{4}$

$= \frac{\pi - \ln 4}{4}$ unit^2

iii) $V_x = \pi\int_0^{\pi/4}(1 - \tan x)^2\,dx$

$= \pi\int_0^{\pi/4}(1 - 2\tan x + \tan^2 x)\,dx$

$= \pi\int_0^{\pi/4}(\sec^2 x - 2\tan x)\,dx$

$= \pi\left[\tan x + 2\ln(\cos x)\right]_0^{\pi/4}$

$= \pi\left[\left(\tan\frac{\pi}{4} + 2\ln\left(\cos\frac{\pi}{4}\right)\right) - (0 + 2\ln 1)\right]$

$= \pi\left[1 + 2\ln\frac{1}{\sqrt{2}}\right]$

or may tidy up to

$= \pi\left[1 + 2\ln 2^{-\frac{1}{2}}\right]$

$= \pi\left[1 - \ln 2\right]$ units^3