Name:_____

Class:_____

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12

HSC ASSESSMENT TASK 3

JUNE 2012

MATHEMATICS Extension 1

Time Allowed: 70 minutes

Instructions:

- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start **each** question on a **new page.**
- Standard integrals can be found on the last page.
- Write in blue or black pen only

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total
/10	/10	/10	/10	/10	/10	/60

Question 1 **(10 marks)** **Marks**

a) Find the exact value of 1

$$\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$$

b) Differentiate the following

(i) $y = x\,e^{x^2}$ 2

(ii) $y = cos^{-1}\,3x$ 2

(iii) $y = \log_e \frac{x+4}{x^2}$ 2

(iv) $y = \log_a x$ 1

c) Simplify $\log_a x^4 \div \log_a x^{\frac{2}{3}}$ 2

Question 2 **(10 marks)** **Marks**

Consider the function $f(x) = \frac{x}{x-3}$

(i) Show that $f'(x) < 0$ for all x in the domain 2

(ii) Find the equation of the horizontal asymptote 1

(iii) Without using any further calculus sketch the graph of $y = f(x)$ showing asymptotes and the x and y intercepts 2

(iv) Explain why $f(x)$ has an inverse function $f^{-1}(x)$ 1

(v) Find an expression for $f^{-1}(x)$ 2

(vi) For what value(s) of x does $f(x) = f^{-1}(x)$? 2

Question 3 **(10 marks)** **Marks**

a) Sand is poured into a conical heap at a constant rate of $0.6\ m^3\ /s$ so that the height of the heap is always equal to twice the radius of the base.

(i) Show that $\frac{dV}{dr} = 2\pi r^2$ 1

(ii) When the heap is 5m high, find the rate of increase of the radius of the base. 2

b) Find

(i) $\int \frac{x}{x^2+4}\ dx$ 1

(ii) $\int \frac{1}{x^2+4}\ dx$ 2

(iii) $\int \frac{dx}{\sqrt{1-4x^2}}$ 2

c) Sketch $y = \sin^{-1}(2x-1)$ showing all important features 2

Question 4 **(10 marks)** **Marks**

a) Find m if $x^{m+2} = e^{6lnx}$ where $x > 0$ 2

b) Find the general solution of $2\sin x = -1$ 2

c) Consider the function $y = \frac{lnx}{x}$ where $x > 0$

(i) The function has a maximum turning point in the domain $x > 0$. Find its coordinates. 2

(ii) Find $\lim_{x\to\infty} \frac{lnx}{x}$ 1

(iii) Sketch the curve showing all important features 2

d) Let $\tan^{-1} x = \propto$. Find an expression for $\cos \propto$ in terms of x given that $\propto$ is acute. 1

Question 5 **(10 marks)** **Marks**

a) Evaluate $\int_1^3 \frac{dx}{(1+x)\sqrt{x}}$ using the substitution $u = \sqrt{x}$ or otherwise. 3
Leave your answer in exact form.

b) Find $\int cos^2 \frac{x}{2}\ dx$ 2

c) Given $f(x) = 3\cos^{-1}(sin2x) - 2\sin^{-1}(cos\ 3x)$, show that $f(x)$ is a constant function by finding $f'(x)$ 3

d) Using the table of standard integrals, find the exact value of 2

$$\int_0^{\frac{\pi}{8}} sec2x\ tan2x\ dx$$

Question 6 **(10 marks)** **Marks**

a)

(i) Express $sinx - cos\ x$ in the form $R\sin(x - \propto)$ where $R > 0$ and $\propto$ is acute. 2

(ii) Hence solve $\sin x - \cos x = -1$ over the domain $0 \leq x \leq 2\pi$ 2

b) Assume that at time t years the population $P(t)$ of a town is given by $P(t) = 50000 + Ae^{kt}$ where A and k are constants.

(i) Show that $P(t)$ satisfies the equation $P'(t) = k\ [P(t) - 50000]$ 1

(ii) Given that $P(0) = 70000$, evaluate A 1

(iii) If $P(8) = 150000$, find k correct to 4 decimal places 2

(iv) How many years will it take for the population to reach 200000? (give your answer correct to the nearest integer) 2

Teacher's Name: Student's Name/N°:

2012 Ext. 1 June Ass. Task Solutions

Question 1

a) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = x$

$\cos x = -\frac{\sqrt{3}}{2}$

$\therefore x = \frac{5\pi}{6}$

b)ci) $y = x e^{x^2}$

$y' = x \times 2x e^{x^2} + e^{x^2}$

$= 2x^2 e^{x^2} + e^{x^2}$

cii) $y = \cos^{-1} 3x$

$y' = \frac{-1}{\sqrt{1-(3x)^2}} \times 3$

$= \frac{-3}{\sqrt{1-9x^2}}$

ciii) $y = \log_e \frac{x+4}{x^2}$

$= \log_e (x+4) - \log_e x^2$

$y' = \frac{1}{x+4} - \frac{2x}{x^2}$

$= \frac{1}{x+4} - \frac{2}{x}$

civ) $y = \log_a x$

$= \frac{\log_e x}{\log_e a}$

$y' = \frac{1}{\log_e a} \times \frac{1}{x}$

$y' = \frac{1}{x \log_e a}$

c) $\log_a x^4 \div \log_a x^{\frac{2}{3}}$

$4 \log_a x \div \frac{2}{3} \log_a x$

$= \frac{4}{\frac{2}{3}}$

$= 4 \times \frac{3}{2} = 6$

Question 2

a) (i) $f(x) = \frac{x}{x-3}$

$f'(x) = \frac{(x-3) \times 1 - x \times 1}{(x-3)^2}$

$f'(x) = \frac{-3}{(x-3)^2} < 0$ for all x

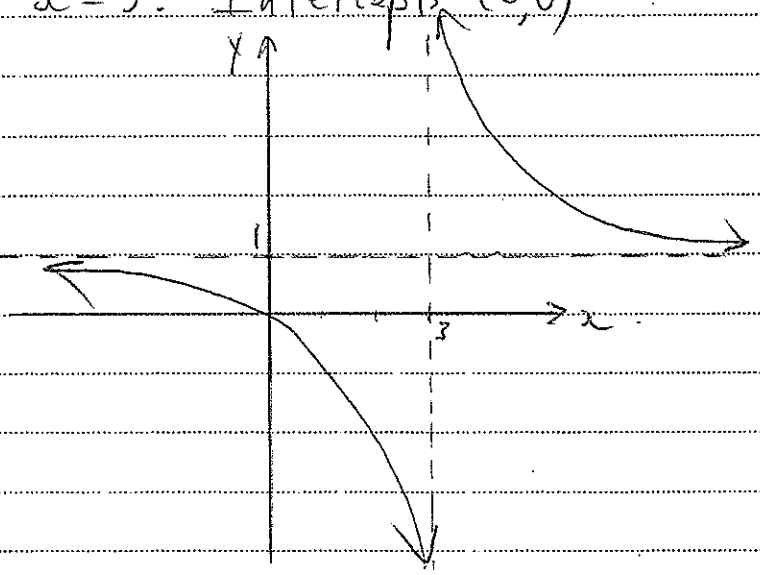
aii) As $x \to \infty$

$f(x) \to 1 \uparrow$

$\therefore y = 1$ is horizontal asymptote

Teacher's Name: **Student's Name/N°:**

(iii) Vertical asymptote $x = 3$. Intercepts $(0,0)$



(iv) Satisfies horizontal line test.

(v) $y = \frac{x}{x-3}$

$x = \frac{y}{y-3}$

$xy - 3x = y$

$xy - y = 3x$

$y(x-1) = 3x$

$y = f^{-1}(x) = \frac{3x}{x-1}$

(vi) $f(x) = f^{-1}(x)$ on $y = x$

$\frac{x}{x-3} = x$

$x = x^2 - 3x$

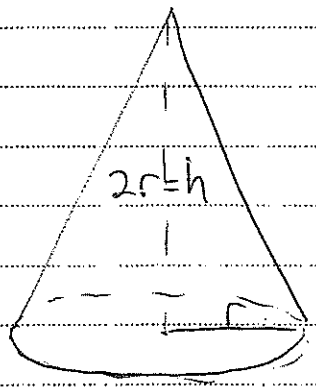
$0 = x^2 - 4x$

$0 = x(x-4)$

At $x = 0$ or 4

Question 3

(a)



$\frac{dV}{dt} = 0.6$

(i) $V = \frac{1}{3}\pi r^2 \times 2r$

$= \frac{2}{3}\pi r^3$

$\frac{dV}{dr} = 2\pi r^2$

(ii) $\frac{dV}{dr} = \frac{dV}{dt} \times \frac{dt}{dr}$

$\frac{dV}{dr} = \frac{dV}{dt} \times \frac{1}{\frac{dr}{dt}}$

$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{1}{\frac{dV}{dr}}$

$= 0.6 \times \frac{1}{2\times\pi} \times 2.5$

$\frac{dr}{dt} = 0.015 \text{ m/s}$

Teacher's Name: **Student's Name/N°:**

b)(i) $\int \frac{2x}{x^2+4}\,dx$

$= \frac{1}{2}\log_e(x^2+4) + c$

(ii) $\int \frac{1}{x^2+4}\,dx$

$= \frac{1}{2}\int \frac{2}{x^2+2^2}\,dx$

$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

(iii) $\int \frac{dx}{\sqrt{1-4x^2}}$

$\int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - x^2)4}}$

$\frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - x^2}}$

$= \frac{1}{2}\sin^{-1}\left(\frac{x}{\frac{1}{2}}\right)$

$= \frac{1}{2}\sin^{-1}(2x) + c$

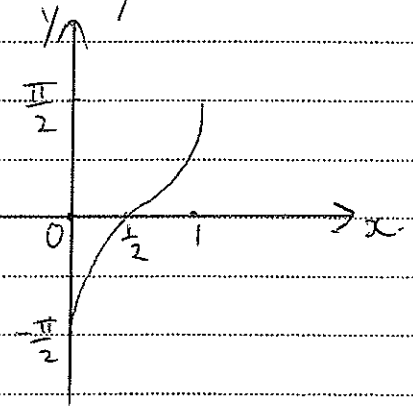
c) $y = \sin^{-1}(2x-1)$

Domain: $-1 \leq 2x-1 \leq 1$

$0 \leq 2x \leq 2$

$0 \leq x \leq 1$

Range: $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$



Question 4

a) $x^{m+2} = e^{6\ln x}$

$x^{m+2} = e^{\ln x^6}$

$x^{m+2} = x^6$

$m = 4$

b) $2\sin x = -1$

$\sin x = -\frac{1}{2}$

$\sin x = \sin\left(\frac{-\pi}{6}\right)$

$x = n\pi + (-1)^n\left(\frac{-\pi}{6}\right)$

where n is an integer

Teacher's Name: **Student's Name/N°:**

c) (i) $y = \frac{\ln x}{x}$

$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \ln x}{x^2}$

$0 = \frac{1 - \ln x}{x^2}$ for stat. pt.

$\therefore 1 - \ln x = 0$

$\ln x = 1$

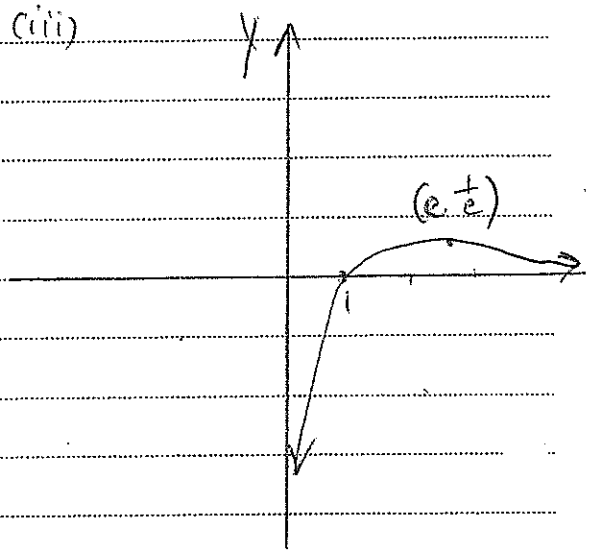
$e^1 = x$

$x = e$

$(e, \frac{1}{e})$ is stat. pt.

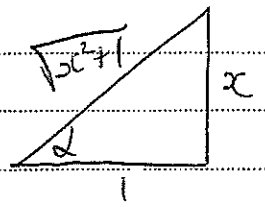
(ii) $\lim_{x \to \infty} \frac{\ln x}{x} = 0$

(iii)



d) $\tan^{-1} x = \alpha$

$\therefore x = \tan \alpha$



$\cos \alpha = \frac{1}{\sqrt{x^2+1}}$

Question 5

a) $\int_1^3 \frac{dx}{(1+x)\sqrt{x}}$

Let $u = \sqrt{x}$

$du = \frac{dx}{2\sqrt{x}}$

$\int_1^{\sqrt{3}} \frac{2u\,du}{(1+u^2)u}$

$2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$

$= \frac{\pi}{6}$

b) $\int \cos^2 \frac{x}{2}\,dx$

$\int \frac{\cos x + 1}{2}\,dx$

$\frac{1}{2}\left[\sin x + x\right]$

$\frac{\sin x}{2} + \frac{x}{2} + C$

Teacher's Name: Student's Name/N°:

c) $f(x) = 3\cos^{-1}(\sin 2x) - 2\sin^{-1}(\cos 3x)$

$$f'(x) = \frac{-3 \times 2\cos 2x}{\sqrt{1-\sin^2 2x}} - \frac{2 \times -3\sin 3x}{\sqrt{1-\cos^2 3x}}$$

$$= \frac{-6\cos 2x}{\cos 2x} + \frac{6\sin 3x}{\sin 3x}$$

$= 0 \quad \therefore f(x)$ must be a constant

d)

$$\int_0^{\frac{\pi}{8}} \sec 2x + \tan 2x \, dx$$

$$= \left[\frac{1}{2}\sec 2x\right]_0^{\frac{\pi}{8}}$$

$$= \frac{1}{2}\sec\frac{\pi}{4} - \frac{1}{2}\sec 0$$

$$= \frac{1}{2\cos\frac{\pi}{4}} - \frac{1}{2\cos 0}$$

$$= \frac{1}{\frac{2}{\sqrt{2}}} - \frac{1}{2}$$

$$= \frac{\sqrt{2}}{2} - \frac{1}{2}$$

$$= \frac{\sqrt{2}-1}{2}$$

Question 6

a) (i) $\sin x - \cos x \equiv A\sin(x-\alpha)$

$A = \sqrt{1^2+1^2}$, $\quad \alpha = \tan^{-1}\frac{1}{1}$

$= \frac{\pi}{4}$

$\therefore \sin x - \cos x = \sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$.

Teacher's Name: Student's Name/N°:

c(i) $\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)=-1$

$\sin\left(x-\frac{\pi}{4}\right)=\frac{-1}{\sqrt{2}}$

$x-\frac{\pi}{4}=\frac{5\pi}{4}\quad\frac{7\pi}{4}$ also $\frac{-\pi}{4}$

$\therefore\ x=0,\ \frac{3\pi}{2},\ 2\pi$

c) ci) $P(t)=50000+Ae^{kt}$

$P'(t)=kAe^{kt}$

$=k(P(t)-50000)$

(ii) $70000=50000+Ae^{kt}$

$20000=Ae^{kt}$

$20000=A$ since $e^0=1$

c(iii) $150000=50000+20000e^{8k}$

$100000=20000e^{8k}$

$5=e^{8k}$

$\log_e 5=8k$

$k=\log_e 5\div 8$

$k=0.2012$ to 4 d.p.

c(iv) $200000=50000+20000e^{0.2012t}$

$150000=20000e^{0.2012t}$

$7.5=e^{0.2012t}$

$\log_e 7.5=0.2012t$

$t=10$ years