Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC COURSE

Extension 1 Mathematics

Assessment 3

June 2013

TIME ALLOWED: 70 minutes

Instructions:

- ***Start each question on a new page.***
- Write your name and class at the top of this page, and on all your answer sheets.
- Hand in your answers attached to the rear of this question sheet.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated within each question are a guide only and may be varied at the time of marking
- It is suggested that you spend no more than 5 minutes on Part A.
- Approved calculators may be used.

PART A: (5 Marks)

Answers to these multiple choice should be completed on the multiple choice answer sheet supplied with your answer booklet.

All questions are worth 1 mark

1	$\frac{d}{dx} ln(\frac{x+1}{2-x}) =$ A. $\frac{3}{(x+1)(2-x)}$ B. $\frac{1-2x}{(x+1)(2-x)}$ C. $\frac{1-x}{(x+1)(2-x)}$ D. $\frac{2x-1}{(x+1)(2-x)}$
2	An indefinite integral of $\frac{1}{2}(e^x + e^{-x})$ is: A. $\frac{1}{2}(e^x + e^{-x})$ B. $-\frac{1}{2}(e^x + e^{-x})$ C. $\frac{1}{2}(e^x - e^{-x})$ D. $-\frac{1}{2}(e^x - e^{-x})$
3	The indefinite integral of $\frac{1}{\sqrt{9-x^2}}$ is: A. $\frac{1}{3}sin^{-1}\frac{x}{3} + k$ B. $sin^{-1}\frac{x}{3} + k$ C. $3sin^{-1}\frac{x}{3} + k$ D. $\frac{1}{3}sin^{-1}3x + k$
4	The value of $sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is: A. $\frac{\pi}{3}$ B. $\frac{\pi}{6}$ C. $-\frac{\pi}{6}$ D. $-\frac{\pi}{3}$
5	The curve y = $\frac{1}{\sqrt{16+x^2}}$ between the lines $x = 0$ and $x = 4$ is rotated about the x-axis. Its volume is given by: A. $\frac{\pi}{16}$ B. $\frac{\pi}{4}$ C. $\frac{\pi^2}{16}$ D. $\frac{\pi^2}{4}$

PART B

QUESTION 1: (8 marks)

Marks

(a) Find $\frac{d}{dx} tan^{-1}\sqrt{x}$ (simplify your answer) **2**

(b) (i) Find $\frac{d}{dx}(e^{-x^2})$ **1**

(ii) Hence find $\frac{d^2}{dx^2}(e^{-x^2})$ **1**

(c) Evaluate $\int_0^{\sqrt{2}} \frac{2}{\sqrt{4-x^2}}\ dx$ **2**

(d) If $y = tan^{-1}x$ find an expression for $\sin 2y$. **2**

QUESTION 2: (Start a new page) (8 marks)

Marks

(a) Find $sin^{-1}\left[cos(\frac{3\pi}{4})\right]$ **1**

(b) You are given the function $f(x) = (x+2)^2$

(i) State the Domain and Range of $f(x)$ and sketch the curve **2**

(ii) Find the largest possible domain of $f(x)$, containing the point (0, 4) for an inverse function $y = f^{-1}(x)$ to exist. **1**

(iii) Find the inverse function $y = f^{-1}(x)$ from part (ii) above and give its Domain and Range **3**

(iv) Sketch $y = f^{-1}(x)$ **1**

QUESTION 3: (Start a new page) (8 marks)

Marks

(a) Find the exact area beneath the curve $y = \frac{e^x}{1+e^x}$, above the x-axis, and between the lines $x = 0$ and $x = 1$ **2**

Give your answer in simplest terms.

(b) A radio-active substance decomposes, and the mass present (M) after t years from a certain date is given by

$M = M_0 e^{-kt}$ where M_0 and k are constants

(i) Show that this is a solution to $\frac{dM}{dt} = -kM$ **1**

(ii) If the initial mass is 100 gm and the mass after 2 years is 80 gm, find the value of k to 2 dec. places. **2**

(iii) Find the number of years taken for the mass to halve (called the *half life* of the substance). Give your answer to 1 decimal place. **2**

(iv) Sketch the graph of $M = M_0 e^{-kt}$ using the vertical axis as M and the horizontal axis as t. ***Show only keypoints.*** **1**

QUESTION 4: (Start a new page) (8 marks)

			Marks
(a)	(i)	Find $\frac{d}{dx}(sin^{-1}x + cos^{-1}x)$	1
	(ii)	Hence find the exact value of $sin^{-1}x + cos^{-1}x$ You must justify your answer NOT just state it.	2
(b)		Evaluate $\int 2^x dx$	1
(c)		The pressure P gm/cm^3 on a mass of gas, of volume vcm^3 is given by the formula $Pv = 1500.$ If the volume is increasing at the rate of $10cm^3/sec$, find the rate at which the pressure is decreasing when the volume is 30 cm^3	4

QUESTION 5 : (Start a new page) (8 marks)

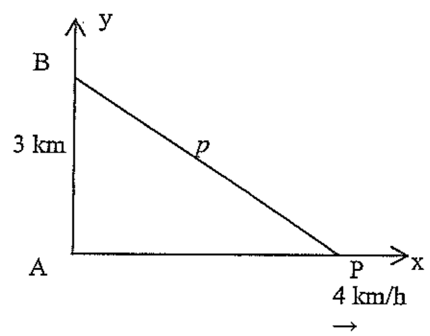
Marks

(a) (i) Find $\frac{d}{dx}(ln(sinx))$ **1**

(ii) Hence, or otherwise, find $\int cot3x\, dx$ **1**

(b) Find $\int \frac{dx}{9+4x^2}$ **2**

(c) A person, P, is walking directly east from a point A at a speed of 4 km/h (ie $\frac{dx}{dt} = 4$) and is being watched by an observer at a point B, which is 3 km due north of A, as shown below: **4**



The distance between the observer and the walker is given as p km.

Find the rate of change of p when P has walked 4 km?

QUESTION 6: (Start a new page) (8 marks)

Marks

(a) Show that the curve $y = \frac{e^{-x}}{1+x^2}$ is decreasing for all x, except $x = -1$ **3**

(b) The population of seals on an island is increasing at a variable rate, and the number of seals (P) at any time t, is given by

$$P = A(1 - e^{-kt}), \text{ where } A \text{ and } k \text{ are constants}$$

(i) Show that $\frac{dP}{dt} = k(A - P)$ **1**

(ii) Show that the maximum seal Population that the island can accommodate is A. **1**

(iii) If one quarter of the maximum population that the island can hold is reached after 5 hours, what fraction is populated after another 5 hours? **3**

End of Examination

SOLUTIONS

PART A

1/ A   2/ C   3/ B   4/ C   5/ C

PART B:

QUESTION 1:

(a) $\dfrac{\frac{1}{2}x^{-\frac{1}{2}}}{1+x} = \dfrac{1}{2\sqrt{x}(1+x)}$ — 1 for the differential; 1 for simplified answer

(b) (i) $dy/dx = -2xe^{-x^2}$ — 1 MARK

(ii) $d^2y/dx^2 = e^{-x^2}(-2) + (-2x)(-2x)e^{-x^2}$ — 1 MARK

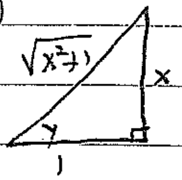
$= -2e^{-x^2}[1-2x^2]$ (or $2xe^{-x^2}(2x^2-1)$)

(c) $2\sin^{-1} x/2 \Big]_0^{\sqrt{2}} = 2\sin^{-1}(1/\sqrt{2}) - 2\sin^{-1}0$ — 1 for the integral

$= 2\pi/4$

$= \pi/2$ — 1 for $\pi/2$

(d)



$\sin 2y = 2\sin y \cos y$

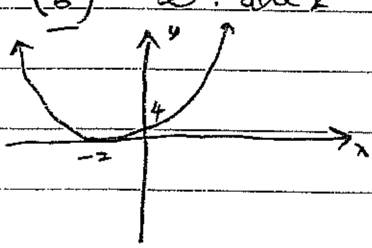
$= 2\dfrac{x}{\sqrt{x^2+1}} \cdot \dfrac{1}{\sqrt{x^2+1}}$ — 1 for each of $\sin y$ and $\cos y$

$= \dfrac{2x}{x^2+1}$

QUESTION 2:

(a) $-\pi/4$ — 1 MARK

(b) $\mathcal{D}$: all x $\mathcal{R}$: $y \geq 0$ — 1 for both $\mathcal{D}$ and $\mathcal{R}$.



1 for the sketch

(ii) $\mathcal{D}_f : x \geq -2$ for f^{-1} to exist ← ①

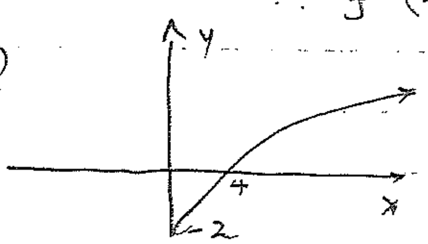
(iii) $y = (x+2)^2 \longrightarrow x = (y+2)^2$

$y = \sqrt{x} - 2$ — 1 MARK for $f^{-1}(x)$

$\therefore f^{-1}(x) = \sqrt{x} - 2$ $\mathcal{D}_{f^{-1}} : x \geq 0$ — 1 MARK

$\mathcal{R}_{f^{-1}} : y \geq -2$ — 1 MARK

(iv)



1 MARK

QUESTION 3:

(a) $\int_0^1 \frac{e^x}{1+e^x}\,dx = \left[\ln(1+e^x)\right]_0^1$ — 1 for $\ln(1+e^x)$

$= \ln(1+e) - \ln(2)$

$= \ln\left[\frac{1+e}{2}\right]$ — 1 for either

(b) (i)

$\frac{dM}{dt} = -kM_0e^{-kt}$ — 1 mark

$= -kM$.

(ii) $M_0 = 100 \Rightarrow M = 100e^{-kt}$

At $t=2$ $M = 80$

$\therefore 80 = 100e^{-2k}$

$\therefore -2k = \ln 0.8$

$\therefore k \approx 0.11$ — 2 marks

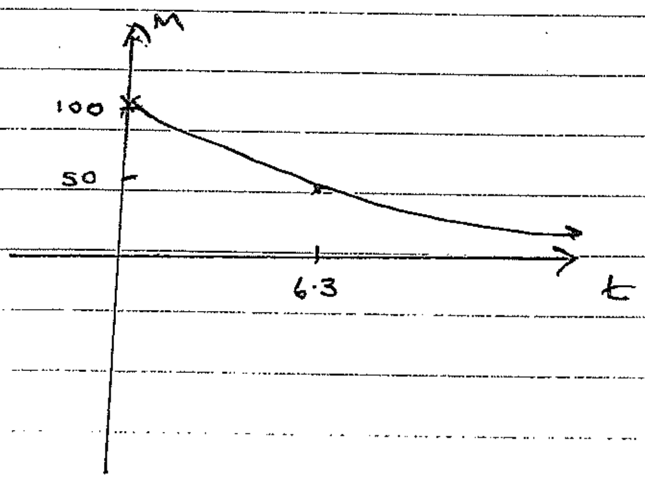
(iii) At $M = 50$

$50 = 100e^{-0.11t}$

$\therefore -0.11t = \ln 0.5$

$t \doteqdot 6.3$ years — 2 marks

(iv)



1 for a "decent" go.

QUESTION 4:

(a) (i) $\frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$ ← (1)

(ii) $\therefore \sin^{-1}x + \cos^{-1}x = c$ a constant since $\frac{d}{dx}(c) = 0$

At $x = 0$: $c = \sin^{-1}(0) + \cos^{-1}(0)$

$= \pi/2$.

$\therefore \sin^{-1}x + \cos^{-1}x = \pi/2$

2 MARKS
1 for finding $\pi/2$ and 1 for a reasonable justification of why.

(b) $\frac{1}{\ln 2}2^x$ (1)

(c) $P = \frac{1500}{v}$

$\therefore dP/dv = -1500/v^2$ and $\frac{dv}{dt} = 10$

$\therefore \frac{dP}{dt} = \frac{dP}{dv} \times \frac{dv}{dt}$

$= \frac{-1500}{v^2} \times \frac{10}{1}$ ← 1 to get here

$= -15000/900$

$= -50/3$ gm/cm^3/sec — 1 MARK

1 MARK for each of $\frac{dv}{dt}$ and $\frac{dP}{dv}$.

QUESTION 5:

(a) (i) $\cos x / \sin x = \cot x$ — 1 for either

(ii) $\int \cot 3x\,dx = \frac{1}{3}\ln(\sin 3x) + k$ — 1 MARK – no penalty for no k

(b) $\frac{1}{4}\int \frac{dx}{9/4 + x^2} = \frac{2}{3}\frac{1}{4}\tan^{-1}\frac{2x}{3} + k$

$= \frac{1}{6}\tan^{-1}2x/3 + k$

2 MARKS
1 only for missing the 1/6

(c) $dx/dt = 4$ $P = \sqrt{9+x^2}$ ← 1 for P.

$dP/dx = \frac{1}{2}\cdot 2x(9+x^2)^{-1/2}$ ← 1 for dP/dx

$dP/dt = dP/dx \cdot dx/dt$

$= \frac{4x}{\sqrt{9+x^2}}$ ← 1 for dP/dt

At $x = 4$: $dP/dt = \frac{16}{5}$ km/hr ← (1)

QUESTION 6:

(a) $\frac{dy}{dx} = \frac{-(1+x^2)e^{-x} - e^{-x} \cdot 2x}{(1+x^2)^2}$

$= \frac{-e^{-x}[1+x^2+2x]}{(1+x^2)^2}$

$= \frac{-e^{-x}(1+x)^2}{(1+x^2)^2} \leq 0$ — 1 mark.

because $(1+x^2) \geq 0$

$(1+x^2)^2 \geq 0$

$e^{-x} \geq 0$

$\forall$ x values — 1 for stating these, or equivalent statements

when $x = -1$ $\frac{dy}{dx} = 0$ — 1 for realising $x = -1$ gives the zero.

$\therefore$ $dy/dx < 0$ $\forall$ x except $x = -1$

$\therefore$ decreasing $\forall$ x " " " "

(b)

(i) $dP/dt = -Ae^{-kt}(-k)$

$= Ake^{-kt}$

$= k(A - A + Ae^{-kt})$

$= k[A - A(1 + e^{-kt})]$

$= k(A - P)$ — 1 for any reasonable attempt

(ii) As $t \to \infty$, $e^{-kt} \to 0$, so $P \to A$ ← ①

This means the population reaches a capacity of A (or the island is "full")

(iii) At $t = 5$, $P = \frac{1}{4}A$ ← ①

$\therefore$ $0.25 = 1 - e^{-5k}$

$\therefore$ $5k = \ln(0.75)$

$k = 0.0575$ — ① for k.

At $t = 10$ $P/A = 1 - e^{-0.575}$

$= 0.4372$

$\approx 44\%$. — ① for 44%, 43.7% or equivalent.