Name:________________________

Class:________________________

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12

HSC ASSESSMENT TASK 3

JUNE 2014

MATHEMATICS Extension 1

Time Allowed: 70 minutes

Instructions:

- Write your name and class at the top of each page.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start **each** question on a **new page.**
- Standard integrals can be found on the last page.

1. What is the derivative of $y = \cos^{-1}(\frac{1}{x})$ with respect to x?

(A) $\frac{-1}{\sqrt{x^2-1}}$

(B) $\frac{-1}{x\sqrt{x^2-1}}$

(C) $\frac{1}{\sqrt{x^2-1}}$

(D) $\frac{1}{x\sqrt{x^2-1}}$

2. The number N of animals in a population at time t years is given by $N=100 + Ae^{kt}$ for constants $A > 0$ and $k > 0$. Which of the following is the correct differential equation?

(A) $\frac{dN}{dt} = k(N-100)$

(B) $\frac{dN}{dt} = -k(N+100)$

(C) $\frac{dN}{dt} = -k(N-100)$

(D) $\frac{dN}{dt} = k(N+100)$

3. If $f(x) = 1 - \cos\frac{x}{2}$ what is the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x) = 2\cos^{-1}(1-x)$

(B) $f^{-1}(x) = \frac{1}{2}\cos^{-1}(1-x)$

(C) $f^{-1}(x) = \frac{1}{2}\cos^{-1}(1+x)$

(D) $f^{-1}(x) = 2\cos^{-1}(1+x)$

4. What is the domain and range of $y = \cos^{-1}(\frac{3x}{2})$?

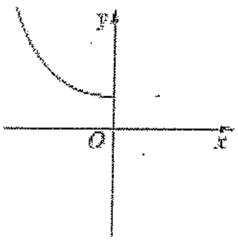
(A) Domain: $-\frac{2}{3} \le x \le \frac{2}{3}$. Range: $0 \le y \le \pi$

(B) Domain: $-1 \le x \le 1$. Range: $0 \le y \le \pi$

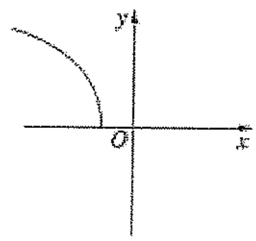
(C) Domain: $-\frac{2}{3} \le x \le \frac{2}{3}$. Range: $-\pi \le y \le \pi$

(D) Domain: $-1 \le x \le 1$. Range: $-\pi \le y \le \pi$

5. The diagram of the graph $y = f(x)$



Which diagram shows the graph of $y = f^{-1}(x)$?

(A)


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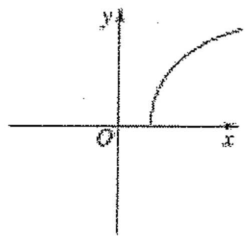
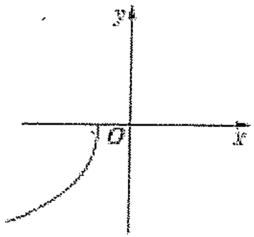


(C)


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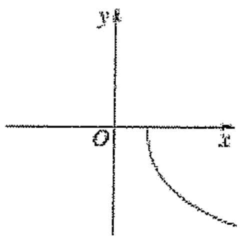


Question 6 (8 marks)

a) Write the exact value of :

i) $sin^{-1}\frac{\sqrt{3}}{2}$ 1

ii) $sin^{-1}(\sin(-\frac{\pi}{4}))$ 1

b) Simplify $\cos\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)$ 2

c) Write the equation $\ln x + \ln y^2 = 3$ without logarithms 1

d) Solve for x: $log_{10}(x^2) + log_{10}x = 1$ 1

e) Find $\frac{d^2}{dx^2}(e^{x^2})$ 2

Start a new page

Question 7 (8 marks)

a) Find the derivative of $\sin^{-1}x + \cos^{-1}x$ 1

and hence find the exact value of $\sin^{-1}x + \cos^{-1}x$

(Show all working) 2

b) Differentiate the following with respect to x:

i) $g(x) = \ln x^2 - e$ 1

ii) $h(x) = \ln(\frac{e^x - 1}{e^x + 1})$ 2

(leaving your answer in simplified exact form)

iii) $y = cos^{-1}(-x) + cos^{-1}(x)$ 2

Start a new page

Question 8 (9 marks)

a) Sketch the curve $y = \sin^{-1} 3x$. 2

b) Differentiate $e^{\tan^{-1} x}$ with respect to x 1

c) i) Find $\frac{d}{dx}\left(xe^x - e^x\right)$ 1

ii) Hence, or otherwise, find $\int_0^1 xe^x \, dx$ 2

d) Find the inverse function for $g(x) = \sqrt{5-x} - 1$ and state the domain and range for the inverse 3

Start a new Page

Question 9 (8 marks)

a) Find the equation of the tangent to the curve $y = 4\sin^{-1}(\frac{x}{2})$ at the point where x = 1. (Leave in exact form) 3

b) Find $\int \frac{\ln 2x}{x} dx$ using the substitution u = ln 2x, or otherwise 2

c) Find the exact value of $\cos\left(\sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{4}{5}\right)\right)$ 3

Start a new page

Question 10 (8 marks)

a) Differentiate $tan^{-1} e^{2x}$ and hence find $\int_0^{\frac{1}{2}} \frac{4e^{2x}}{1+e^{4x}} \, dx$ as an exact answer 3

b) The rate at which a body cools in air is proportional to the difference between the temperature, T, of the body and the constant surrounding temperature, S. this can be expressed as $\frac{dT}{dt} = k(T - S)$ where t is time in minutes and k is a constant.

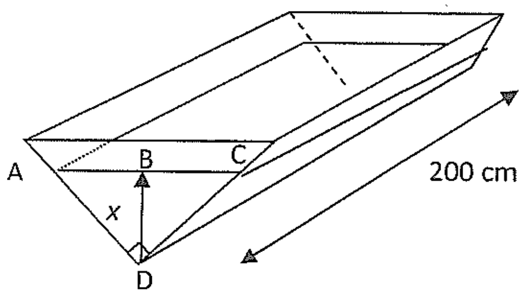
i. Show that $T = S + Be^{kt}$ where B is a constant, is a solution of the above equation 1

ii. If a particular body cools from 100^0 to 80^0 in 30 minutes, find the temperature of the body after a further 30 minutes, given the surrounding temperature remains a constant 25^0. Give your answer to the nearest degree. 4

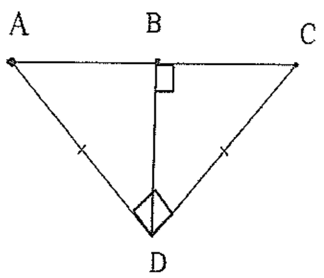
Start a new page

Question 11 (9 marks)

a) A water trough is 200 cm long and has the cross section of a right-angled isosceles triangle. B is the midpoint of the line AC. 'x' is the depth of the water in the trough.



(i)



Prove that ~~AD = DC~~ BO = BC. 2

(ii) Show that when the depth of the water is x cm, the volume of the water in the tank is $200x^2$ cm^3, explaining all steps. 1

(iii) Water is poured in at a constant rate of 5 litres per minute. Find the rate at which the water level is rising when the depth is 30 cm (1 litre = 1000 cm^3) 2

b) Differentiate $\left(\tan^{-1}\left(\frac{x}{3}\right)\right)^2$, and hence find the exact value of $\int_0^{\sqrt{3}} \frac{\tan^{-1}\left(\frac{x}{3}\right)}{x^2+9}\,dx$ 2

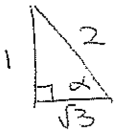
c) By writing $y = tan^{-1}\sqrt{x}$ in the form $x = f(y)$, show that $\frac{dy}{dx} = \frac{1}{2\sqrt{x}\,(1+x)}$ 2

D A A A D

Q6 (a)(i) $\sin^{-1}\frac{\sqrt{3}}{2} = \pi/3$

(ii) $\sin^{-1}\left(\sin -\pi/4\right) = -\pi/4$

(b) $\cos\left(2\cos^{-1}\sqrt{3}/2\right)$



$\cos^{-1}\sqrt{3}/2 = \alpha$

$\cos\alpha = \sqrt{3}/2$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$
$$= 2\left(\sqrt{3}/2\right)^2 - 1$$
$$= 6/4 - 1$$
$$= 1/2$$

(c) $\ln x + \ln y^2 = 3$

$\ln\left(xy^2\right) = 3$

$e^3 = xy^2$

(d) $\log_{10}\left(x^2\right) + \log_{10} x = 1$

$\log_{10} x^3 = \log_{10} 10$

$x^3 = 10$

$x \doteqdot 2\cdot154$

(e) $y = e^{x^2}$

$y = e^u \qquad u = x^2$

$\frac{dy}{du} = e^u \qquad \frac{du}{dx} = 2x$

$\frac{dy}{dx} = 2xe^{x^2}$

$$\frac{d^2y}{dx^2} = 2x\left(2xe^{x^2}\right) + 2e^{x^2}$$
$$= e^{x^2}\left(4x^2 + 2\right)$$
$$= 2e^{x^2}\left(2x^2 + 1\right)$$

Q7 (a) $f(x) = \sin^{-1}x + \cos^{-1}x$

$f'(x) = 0$

$\therefore$ gradient constant

$0 + \pi/2 = C$

$\sin^{-1}x + \cos^{-1}x = \pi/2$.

OR

$\sin^{-1}x = \alpha$. $\cos^{-1}x = \pi/2 -$

$\sin\alpha = x \quad \cos(\pi/2 - \alpha)$

$\sin^{-1}x + \cos^{-1}x = \alpha + \pi/2 - \alpha$

$= \pi/2$.

(b) (i) $g(x) = \ln x^2 - e$

$g'(x) = 2/x$.

(ii) $h(x) = \ln\left(\frac{e^x - 1}{e^x + 1}\right)$

$h(x) = \ln u$

$$= \frac{1}{u}$$
$$= \frac{e^x + 1}{e^x - 1} \cdot \frac{2e^x}{(e^x+1)^2}$$
$$= \frac{2e^x}{(e^x-1)(e^x+1)}$$

$u = (e^x - 1)(e^x + 1)^{-1}$

$u' = (e^x - 1)(-1)(e^x+1)^{-2}(e^x$

$+ e^x(e^x+1)^{-1}$

$$= \frac{-e^x(e^x-1)}{(e^x+1)^2} + \frac{e^x}{e^x+1}$$
$$= \frac{-e^x(e^x-1) + e^x(e^x+1)}{(e^x+1)^2}$$
$$= \frac{2e^x}{(e^x+1)^2}$$

OR $h(x) = \ln(e^x - 1) - \ln(e^x + 1)$

$$= \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1}$$

$$= \frac{e^x(e^x+1) - e^x(e^x-1)}{(e^x-1)(e^x+1)} = \frac{2e^x}{(e^x-1)(e^x+1)}$$

Q7 (b)(iii) $y = \cos^{-1}(-x) + \cos^{-1}(x)$

$y' = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$

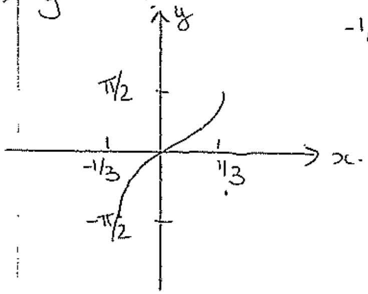
$= 0.$

Q8 (a) $y = \sin^{-1} 3x.$

$-1 \le 3x \le 1$

$-1/3 \le x \le 1/3$

$-\pi/2 \le y \le \pi/2$



(b) $y = e^{\tan^{-1} x}$

$\frac{dy}{dx} = e^u \qquad u = \tan^{-1} x.$

$= e^u \qquad u' = \frac{1}{1+x^2}$

$\frac{dy}{dx} = \frac{e^{\tan^{-1} x}}{1+x^2}$

(c) (i) $\frac{d}{dx}\left(xe^x - e^x\right) = e^x + xe^x - e^x$

$= xe^x$

(ii) $\int_0^1 xe^x\,dx = \left[xe^x - e^x\right]_0^1$

$= (e^1 - e^1) - (0-1)$

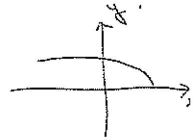
$= 1$

(d)

Q8 (d) $y = \sqrt{5-x} - 1$

D: $x \le 5$

R: $y \ge -1$



$x = \sqrt{5-y} - 1$

$x+1 = \sqrt{5-y}$

$(x+1)^2 = 5-y$

$(x+1)^2 - 5 = -y$

$y = 5 - (x+1)^2$

$= 5 - x^2 - 2x - 1$

$= -x^2 - 2x + 4$

D: $x \ge -1$ R: $y \le 5$

Q9 (a) $y = 4\sin^{-1}\left(\frac{x}{2}\right)$

$y' = \frac{4}{\sqrt{2^2 - x^2}}$

$= \frac{4}{\sqrt{4-x^2}}$

at $x = 1$

$m = 4/\sqrt{3}$

$= \frac{4\sqrt{3}}{3}$

at $x=1$ $y = 4\sin^{-1}(1/2)$

$= 2\pi/3$

$y - 2\pi/3 = \frac{4\sqrt{3}}{3}(x-1)$

$3y - 2\pi = 4\sqrt{3}(x-1)$

$4\sqrt{3}x - 4\sqrt{3} + 2\pi - 3y = 0$

$4\sqrt{3}x - 3y + 2(\pi - 2\sqrt{3}) = 0$

(b) $\int \frac{\ln 2x}{x}\,dx$

$u = \ln 2x$

$du = 1/x\,dx.$

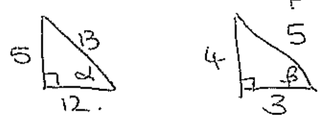
$= \int u\,du$

$= \frac{u^2}{2} + c$

$= \frac{(\ln 2x)^2}{2} + c$

(c) $\cos\left(\sin^{-1}(5/13) + \sin^{-1}(4/5)\right)$

let $\alpha = \sin^{-1} 5/13$ let $\beta = \sin^{-1} 4/5$



$= \cos(\alpha + \beta)$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$=$ 16/65

Q10 (a) $\tan^{-1} e^{2x}$

$y = \tan u \qquad \frac{dy}{du} = \frac{1}{1+u^2}$

$u = e^{2x} \qquad \frac{du}{dx} = 2e^{2x}.$

$\frac{dy}{dx} = \frac{2e^{2x}}{1+(e^{2x})^2}$

$= \frac{2e^{2x}}{1+e^{4x}}$

(ii) $\int_0^{1/2} \frac{4e^{2x}}{1+e^{4x}} = 2\left[\tan^{-1} e^{2x}\right]_0^{1/2}$

$= 2\left[\tan^{-1} e - \tan^{-1} 1\right]$

$= 2\left(\tan^{-1} e - \pi/4\right)$

$= 2\tan^{-1} e - \pi/2$

(b) (i) $\frac{dT}{dt} = k(T-S)$

(i) $T = S + Be^{kt}$

$\frac{dT}{dt} = k\left(Be^{kt}\right)$

$= k(T-S)$

(ii) $T_i = 100 \qquad T_{30} = 80 \qquad t = 30\text{mins.} \qquad S = 25$

$100 = 25 + Be^{0}$

$B = 75$

$80 = 25 + 75e^{k30}$

$11/15 = e^{k30}$

$k = -0.0103$

or $\frac{\ln 11/15}{30}$

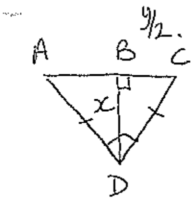
$T_{60} = \qquad t = 60 \qquad S = 25$

$T = 25 + 75e^{(-0.0103 \times 60)}$

$= 65.33^{\circ} \qquad = 65^{\circ}$

Q11

(i)



ΔACD is right angled isosceles

$\therefore \angle DAC = \angle DCA$

$\angle DAC = \angle DCA = 45^{\circ}$ (angle sum of triangle equals 180°)

$\angle ADC = 90^{\circ}$ (given)

(ii) $\tan 45^{\circ} = \frac{x}{y/2}$

$1 = x/y/2$

$\frac{y}{2} = x$

$y = 2x.$

$V = \frac{1}{2}bh \times 200$

$= \frac{1}{2}(2x)(x)(200)$

$V = 200x^2$

$\frac{dV}{dt} = 200 \quad 2x\frac{dx}{dt}$

$\frac{dV}{dt} = 5l$

$= 5000\ cm^3$

at $x = 30$

$5000 = (200)(2)(30)\frac{dx}{dt}$

$\frac{dx}{dt} = 5/12\ cm^3/min$

(b)(i) $y = \left(\tan^{-1}\left(\frac{x}{3}\right)\right)^2$

$f'(x) = 2\tan^{-1}\left(\frac{x}{3}\right)\frac{3}{9+x^2}$

$= \frac{6\tan^{-1}(x/3)}{9+x^2}$

(ii) $\int_0^{\sqrt{3}} \frac{\tan^{-1}\left(\frac{x}{3}\right)}{x^2+9}\,dx = \frac{1}{6}\int \frac{6\tan^{-1}\left(\frac{x}{3}\right)}{x^2+9}\,dx$

$= 1/6\left[\left(\tan^{-1}\frac{x}{3}\right)^2\right]_0^{\sqrt{3}}$

$= 1/6\left[\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)^2 - \left(\tan^{-1} 0\right)^2\right]$