Name: .... Maths Teacher: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Mathematics Extension 1

HSC Course

Assessment 3

June, 2016

Time allowed: 90 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A BOSTES Reference Sheet is provided with this Examination. Please do not write on it.

Section 1 Multiple Choice
Questions 1-5
5 Marks

Section II Questions 6-11
60 Marks

PART A: (5 Marks) ***Use the multiple choice answer sheet at the front of your Answer Booklet.***

All questions are worth 1 mark

1	$tan^{-1}(\sqrt{3}) =$ A. $\frac{\pi}{6}$ B. $\frac{\pi}{3}$ C. $-\frac{\pi}{3}$ D. $-\frac{\pi}{6}$
2	$log_8 128 =$ A. e^{128} B. 16 C. $\frac{3}{7}$ D. $\frac{7}{3}$
3	$\sin(cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)) =$ A. $-\frac{1}{2}$ B. $\frac{1}{2}$ C. $-\frac{\pi}{6}$ D. $\frac{\pi}{6}$
4	$\frac{d}{dx}\,(log_2 3x) =$ A. $\frac{3}{xln2}$ B. $\frac{3ln2}{x}$ C. $\frac{1}{xln2}$ D. $\frac{ln2}{x}$
5	$\tan(cos^{-1}x) =$ A. $\frac{\sqrt{1-x^2}}{x}$ B. $\frac{-\sqrt{1-x^2}}{x}$ C. $\frac{x}{\sqrt{1-x^2}}$ D $\frac{-x}{\sqrt{1-x^2}}$

PART B

(START EACH QUESTION ON A NEW PAGE)

QUESTION 6: (10 Marks)

Marks

(a) Find indefinite integrals of: **3**

(i) $\frac{1}{x^2+9}$ (ii) $\frac{1}{1-3x}$ (iii) tan^2x

Find: (i) $\frac{d}{dx}(\frac{lnx}{x})$ (ii) $\frac{d}{dx}\ e^{sinx}$ **2**

(c) Find the exact value of $\int_0^2 \frac{2\ dx}{\sqrt{4-x^2}}$ **2**

(d) Find the second derivative of e^{x^3} **3**

QUESTION 7: (10 Marks) Start a New Page

Marks

(a) (i) Find the largest Domain, containing the point (4, 4) for which $f(x) = (x-2)^2$ has an inverse function. **1**

(ii) Sketch $y = f^{-1}(x)$ where $f^{-1}(x)$ is the inverse function defined in part (i) **2**

(iii) State the Domain and Range of $f^{-1}(x)$ **2**

(b) Sketch the graph of $y = 2cos^{-1}\frac{x}{2}$ **2**

(c) (i) Using one set of axes, neatly sketch the graphs of $y = e^x$ and $y = e^{-x}$. **3**

(ii) On the same set of axes, use part (i) to sketch $y = \frac{1}{2}(e^x + e^{-x})$

(clearly label this graph)

QUESTION 8: (10 Marks) Start a New Page

Marks

(a) Solve for x: $ln(x+1) = 5$, giving your answer correct to 3 dec. places **1**

(b) The area under $y = \frac{1}{\sqrt{4+x^2}}$ and between the lines $x = 0$ and $x = 2\sqrt{3}$ is rotated about the x-axis. **3**

Find the exact volume of the solid formed.

(c) (i) Find $\frac{dy}{dx}$ if $y = x tan^{-1}x$ **1**

(ii) Hence show that $\int_0^1 tan^{-1}x \, dx = \frac{\pi}{4} - \frac{1}{2}\ln 2$ **3**

(d) (i) Find $\frac{d}{dx}\{ sin^{-1}x + cos^{-1}x\}$ **1**

(ii) What does this imply about the value of the expression $sin^{-1}x + cos^{-1}x$ as x varies over the Domain $-1 \leq x \leq 1$? **1**

QUESTION 9: (10 Marks) Start a New Page

Marks

(a) Show that $\frac{d}{dx} ln\left(\frac{\sqrt{x-1}}{x}\right) = \frac{2-x}{2x(x-1)}$ **2**

(b) The radius of a balloon which is deflating slowly, is decreasing at a rate of 2 cm per minute. **3**

At what rate is the volume decreasing when the radius is 10 cm?

(Give your answer in terms of π)

(**NOTE:** *The volume of a sphere is given by* $V = \frac{4}{3}\pi r^3$)

(c) Find $\int \frac{1}{\sqrt{9-4x^2}}\, dx$ **2**

(d) Show that $tan^{-1} \propto + tan^{-1}\beta = tan^{-1}(\frac{\propto + \beta}{1-\alpha\beta})$ **3**

For $0 < \alpha < 1$ and $0 < \beta < 1$

QUESTION 10: (10 Marks) Start a New Page

(a) Find the derivative of 5^x **1**

(b) (i) Show that $1 + \frac{1}{2x-1} = \frac{2x}{2x-1}$ **1**

(ii) Hence find $\int \frac{x}{2x-1}\,dx$ **2**

(c) Find $\int \cot x\,dx$ **2**

(d) (i) Find $\frac{d}{dx} ln\{x + \sqrt{x^2-1}\}$ **2**

(ii) Hence, or otherwise, find $\int_1^3 \frac{1}{\sqrt{x^2-1}}\,dx$, leaving your answer in exact form. **2**

QUESTION 11: (10 Marks) Start a New Page

Marks

You are given the curve $y = \frac{x}{x^2+1}$

(i) Show that this is an odd function. **1**

(ii) Show that the curve has turning points at $(1, \frac{1}{2})$ and $(-1, -\frac{1}{2})$ and describe their nature **3**

You do not need to find any inflexion points.

(ii) Evaluate $lim \; (\frac{x}{x^2+1})$ **1**

(iii) Sketch the curve, showing all the information you have just found above **2**

(iv) Find the area under this curve and between the lines $x = 1$ and $x = 2$. **1**

(v) With reference to your sketch, and your answer to part (iv) above, explain why **2**

$$\ln\left(\frac{5}{2}\right) < 1$$

END OF THE EXAMINATION

EXT 1 2016 JULY

SOLUTIONS

1/ B 2/ D 3/ B 4/ C 5/ A

Question 6:

(a) (i) $\frac{1}{3}\tan^{-1}\frac{x}{3} + k$ (ii) $-\frac{1}{3}\ln(1-3x) + k$ — 1 mark each (no penalty for no "k")

(iii) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$

$= \tan x - x + k$

(b) (i) $\frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$ — 1 for either

(ii) $\cos x \, e^{\sin x}$ — 1 MARK

(c) $2\sin^{-1}\frac{x}{2}\Big]_0^2 = 2\sin^{-1}1 - 2\sin^{-1}0$ — 1 for integral

$= \pi$ — 1 MARK

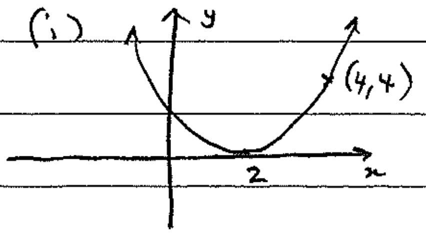
(d) $y' = 3x^2 e^{x^3}$ — 1 MARK

$y'' = e^{x^3}.6x + 3x^2.3x^2 e^{x^3}$

$= 3xe^{x^3}\left[2 + 3x^3\right]$ — 2 MARKS

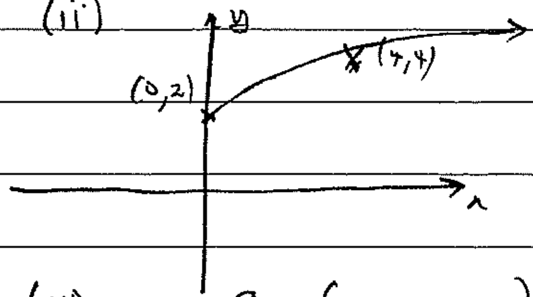
QUESTION 7:

(a) (i)



$\mathcal{D}: x \geq 2$ — 1 MARK

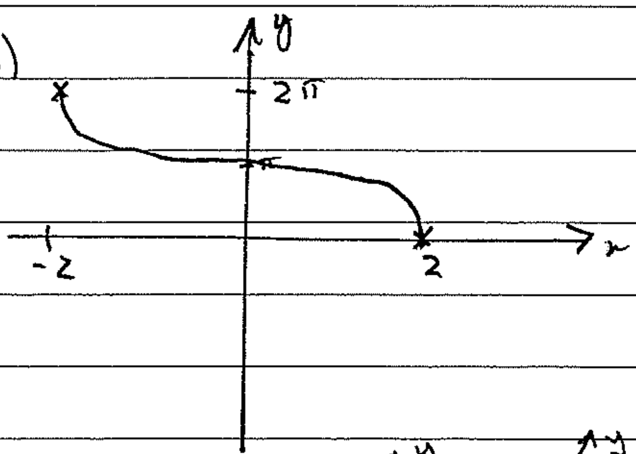
(ii)



1 for shape
1 for (0,2)

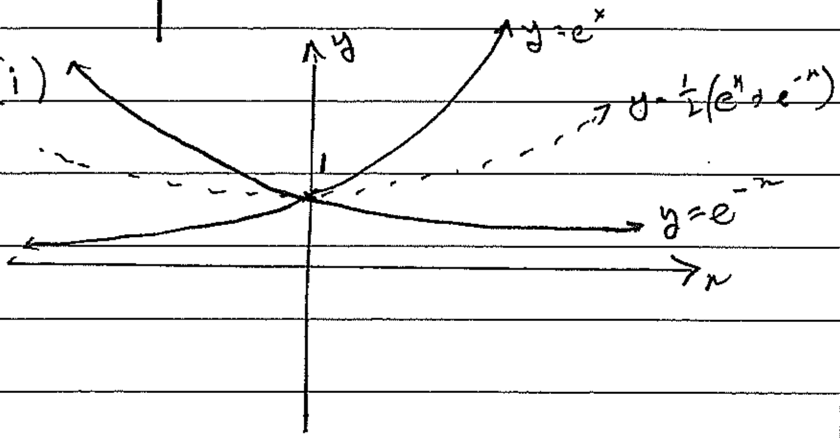
(iii) $\mathcal{D}: \{x : x \geq 0\}$ $\mathcal{R}: \{y : y \geq 2\}$ — 1 each = 2

(b)



KEYPOINTS for 2 MARKS:
(i) (−2,0) and (2,0)
(ii) (0, 2π)
(iii) shape

(c) (i)



1 for each graph = 3

QUESTION 8:

(a) $x + 1 = e^5$

$x = e^5 - 1$

≈ 147.413

1 MARK (MUST be to 3 dec places)

(b) $\text{VOL} = \pi \int_0^{2\sqrt{3}} \frac{1}{4+x^2}\,dx$

1 for correct integral

$= \pi \frac{1}{2}\left[\tan^{-1} x/2\right]_0^{2\sqrt{3}}$

$= \pi/2 \cdot \tan^{-1}\sqrt{3}$

$= \pi^2/6$

2 MARKS

(c) (i) $\frac{dy}{dx} = \tan^{-1}x \cdot 1 + x \cdot \frac{1}{1+x^2}$

1 MARK

(ii) $\int_0^1 \tan^{-1}x\,dx = \int_0^1 \left(\tan^{-1}x + x/1+x^2 - x/1+x^2\right)dx$

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\Big]_0^1$

$= \tan^{-1}(1) - \frac{1}{2}\ln(2)$

$= \pi/4 - \frac{1}{2}\ln 2$

① for realising the connection
① for integrals
② for answer

(d) (i) $\frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$

1 MARK

(ii) This means that it is a constant.

The value is $\pi/2$

1 for "constant"

QUESTION 9:

(a) $\frac{dy}{dx} = \frac{d}{dx}\ln\sqrt{x-1} - \frac{d}{dx}\ln x$

$= \frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{x}$ ← 1 for this

$= \frac{x - 2(x-1)}{2x(x-1)}$

$= \frac{2-x}{2x(x-1)}$ 1 for simplification

(b) $\frac{dr}{dt} = 2$ $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$ ← 1 MARK

$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$ ← 1 MARK

$= 4\pi r^2 . 2$

At $r = 10$ $\frac{dV}{dt} = 800\pi$ 1 MARK

(c) $\int \frac{1/3}{\sqrt{1-\frac{4}{9}x^2}}\,dx$ OR $\int \frac{1/2}{\sqrt{9/4 - x^2}}\,dx$ 1 for arranging $\int$

$= \frac{3}{2}\cdot\frac{1}{3}\sin^{-1}\frac{2x}{3} + k$ $= \frac{1}{2}\sin^{-1}\frac{x}{3/2}$ 1 for answer.

$= \frac{1}{2}\sin^{-1}\frac{2x}{3}$ $= \frac{1}{2}\sin^{-1}\frac{2x}{3}$

(d) Let $A = \tan^{-1}\alpha$ and $B = \tan^{-1}\beta$.

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\alpha+\beta}{1-\alpha\beta}$ 3 MARKS.

$\therefore A + B = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$

QUESTION 10:

(a) $\frac{d}{dx}(5^x) = (\ln 5)5^x$ — 1 MARK

(b) (i) $1 + \frac{1}{2x-1} = \frac{2x-1+1}{2x-1}$ — 1 MARK

$= 2x/2x-1$

(ii) $\int 2/2x-1 \, dx = \frac{1}{2}\int 2x/2x-1 \, dx$

$= \frac{1}{2}\int 1 + \frac{1}{2x-1} \, dx$ ← 1 for this

$= x/2 + \frac{1}{4}\ln(2x-1) + k$ — 1 MARK (no penalty for no "k")

(c) $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$ ← 1 for realising

$= \log \sin x + k$ ← 1 for answer

(d) (i) $\frac{1 + \frac{1}{2} \cdot 2x(x^2-1)^{-1/2}}{x + \sqrt{x^2-1}}$ ← 1 MARK

$= \frac{(x^2-1)^{-1/2}\left[(x^2-1)^{1/2} + x\right]}{x + \sqrt{x^2-1}}$

$= (x^2-1)^{-1/2}$

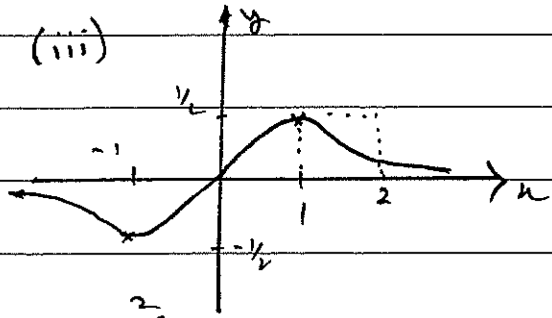
$= \frac{1}{\sqrt{1-x^2}}$ } 1 MARK

(ii) $\int_1^3 \frac{1}{\sqrt{x^2-1}} \, dx = \ln\{x + \sqrt{x^2-1}\}\Big]_1^3$ ← 1 for realising

$= \ln(3+\sqrt{8}) - \ln\{1+0\}$

$= \ln(3+\sqrt{8})$ or $\ln(3+2\sqrt{2})$ } 1 for either answer

QUESTION 11:

(i) $f(a) = \frac{a}{a^2+1}$	1 MARK
$f(-a) = \frac{-a}{(-a)^2+1}$	
$= -f(a)$ $\therefore$ ODD	
(ii) $dy/dx = \frac{(x^2+1)1 - x.2x}{(x^2+1)^2}$	
$= \frac{1-x^2}{(x^2+1)^2}$	
At T.P.'s $dy/dx = 0$	
$\therefore \begin{cases} x = 1 \\ y = 1/2 \end{cases}$ or $\begin{cases} x = -1 \\ y = -1/2 \end{cases}$	1 for each point = 2
x: 0, 1, 2; y'': +, 0, − / x: −2, −1, 0; y': −, 0, +	1 for identification
$\therefore$ MAX at $(1, 1/2)$ min at $(-1, -1/2)$	
(iii) $\lim_{x \to \infty} \left(\frac{x}{x^2+1} \right) = 0$	1 MARK
(iii) 	2 MARKS KEYPOINTS: both T.P.'s both asymptotes.
(iv) $\int_1^2 \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)\Big]_1^2$	
$= \frac{1}{2}\ln(5) - \frac{1}{2}\ln(2)$ $= \frac{1}{2}\ln(5/2)$	1 for either
(v) The area of the dotted rectangle above is $\frac{1}{2}\times 1$ which is greater than the actual area	1 MARK
ie $\frac{1}{2} > \frac{1}{2}\ln(5/2)$	1 MARK

$\therefore \ln(5/2) < 1$