YEAR 12 EXTENSION 2 MATHEMATICS JUNE 2006

42

NAME		RESULT	

DIRECTIONS	▪ Full working should be shown in every question. Marks may be deducted for careless or badly arranged work. ▪ Use black or blue pen only (*not pencils*) to write your solutions. ▪ No liquid paper is to be used. If a correction is to be made, one line is to be ruled through the incorrect answer.

Time allowed : 1 hour 10 minutes

Question 1

a) Find

(i) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ 2

(ii) $\int \frac{dx}{x^2+4x+13}$ 2

(iii) $\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$ 4

b) Resolve $\frac{8}{(x+2)(x^2+4)}$ into partial fractions and hence find $\int \frac{8}{(x+2)(x^2+4)}\,dx$ 4

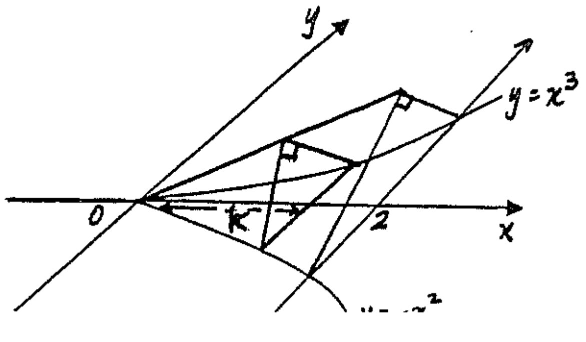
c) Evaluate $\int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\sin\theta}$ using the substitution $t=\tan\frac{\theta}{2}$ 4

Question 2 (Start a new page)

a) The base of a solid is the region in the xy plane enclosed by the curves $y=x^3$, $y=-x^2$ and the line $x=2$. Each cross-section perpendicular to the x-axis is a right-angled isosceles triangle with the hypotenuse in the base of the solid.

(i) Show that the area of the triangular cross-section at $x=k$ is $\frac{1}{4}\left(k^4+2k^5+k^6\right)$. 2

(ii) Hence find the volume of the solid. 2



Question 2 (continued)

b) The area under the curve $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{3}$ is rotated about the y axis to form a solid. 4

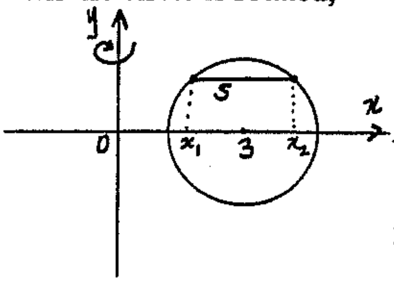
Use the method of cylindrical shells to find the volume of the resulting solid of revolution.

c) Given $\int_{-a}^{a} f(x)dx = \int_{0}^{a}[f(x)+f(-x)]dx$, use this result to find 4

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{e^x}{1+e^x}\tan^2 x\,dx$$

Question 3 (Start a new page)

a) The circle $(x-3)^2 + y^2 = 4$ is rotated about the y-axis to form a ring. When the circle is rotated, the line segment S at height y sweeps out an annulus.



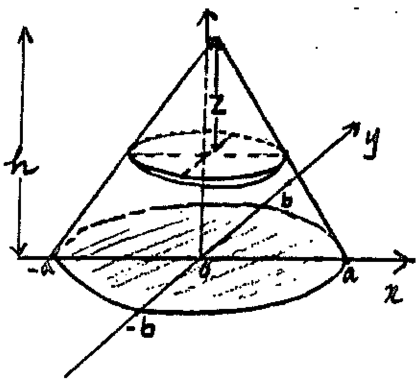
The x coordinates of the endpoints of S are x_1 and x_2.

(i) Show that the area of the annulus is equal to $12\pi\sqrt{4-y^2}$. 2

(ii) Hence find the volume of the ring. 3

b) The solid is an elliptical cone of height h standing on a base with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Consider a slice of width δz parallel to the base at a distance z from the apex of the cone, and similar to the base.

You may assume that the area of an ellipse is given by πab where $2a$ and $2b$ are the lengths of the major and minor axes.



(i) Show that the area of the cross-sectional area of the slice is $\frac{\pi abz^2}{h^2}$. 3

(ii) Find the volume of the elliptical cone. 2

c) $I_n = \int_0^1 \frac{x^n}{1+x^2}dx$ where $n = 0, 1, 2……$

(i) Evaluate I_0 1

(ii) Prove that $I_n + I_{n+2} = \frac{1}{n+1}$ 2

(iii) Show that $I_8 = \frac{1}{7}-\frac{1}{5}+\frac{1}{3}-1+\frac{\pi}{4}$ 1

16 + 14 + 14 = 42 TOTAL

QUESTION 1 16 marks JUNE 2006 Ext 2

a) (i) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ Let $u = \sqrt{x}$, $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$= \int 2\sin u\,du$ ✓

$= -2\cos u + C$ ✓

$= -2\cos\sqrt{x} + C$ (−1) if not substituted back

(ii) $\int \frac{dx}{x^2+4x+13}$

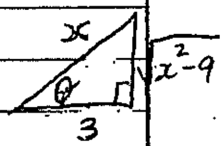
$= \int \frac{dx}{(x+2)^2+3^2}$ ✓

$= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C$ ✓

(iii) $\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$ $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$ ✓

$= \int \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\sqrt{9\tan^2\theta}}$ ✓

$= \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta$



$= \frac{1}{9}\int \cos\theta\,d\theta$

$= \frac{1}{9}\sin\theta + C$

$= \frac{1}{9}\frac{\sqrt{x^2-9}}{x}$ ✓

b) $\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$

$8 = A(x^2+4) + (Bx+C)(x+2)$

$8 = (A+B)x^2 + (2B+C)x + 4A+2C$

Matching coeff of x^2: $A+B=0$

$\therefore A=-B$

Coeff of x: $2B+C=0$ ... ①

Constants: $4A+2C = -4B+2C = 8$. ②

$\therefore B=-1,\ C=2$ ✓✓

$A=1$

$\therefore \frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{2-x}{x^2+4}$

$I = \int \frac{1}{x+2} + \frac{2-x}{x^2+4} = \int \frac{1}{x+2} + \frac{2}{x^2+4} - \frac{x}{x^2+4}$

$= \log_e(x+2) + \tan^{-1}\frac{x}{2} - \frac{1}{2}\log(x^2+4)$ ✓✓

c) $t = \tan\frac{\theta}{2}$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2} = \frac{1}{2}\left(\tan^2\frac{\theta}{2}+1\right)$

$\therefore \frac{2\,dt}{t^2+1} = d\theta$ Limits $\theta = \pi/3,\ t = \frac{1}{\sqrt{3}}$; $\theta = 0,\ t = 0$

$\int_0^{\pi/3} \frac{d\theta}{1+\sin\theta} = \int_0^{1/\sqrt{3}} \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2\,dt}{1+t^2}$ ✓

$= \int_0^{1/\sqrt{3}} \frac{1+t^2}{1+t^2+2t} \cdot \frac{2\,dt}{1+t^2}$

$= \int_0^{1/\sqrt{3}} \frac{1}{(t+1)^2}\,dt$ ✓

$= \left[\frac{1}{t+1}\right]_0^{\frac{1}{\sqrt{3}}}$ ✓

$= \frac{\sqrt{3}}{1+\sqrt{3}} - 1$

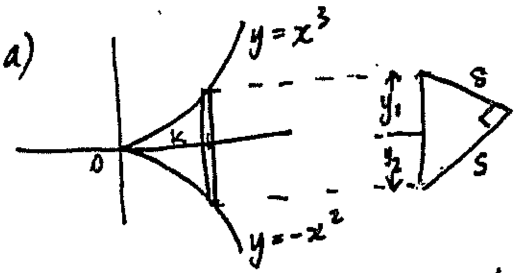
$= \frac{-1}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}$ ①

$= \frac{-1+\sqrt{3}}{1-3}$

$= \frac{\sqrt{3}-1}{2}$

Question 2 (12 marks)

a)



Hypotenuse length $= k^3+k^2$ ✓

$2s^2 = (k^3+k^2)^2$

$= k^6+2k^5+k^4$

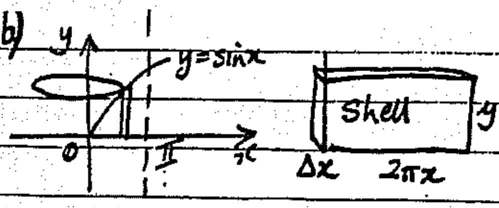
Area of $\Delta = \frac{1}{2}s^2$ ✓

$= \frac{1}{4}(k^6+2k^5+k^4)$

Volume $= \int_0^2 \frac{1}{4}\, k^6+2k^5+k^4\,dk$

$= \frac{1}{4}\left[\frac{k^7}{7} + \frac{k^6}{3} + \frac{k^5}{5}\right]_0^2$ ✓

= 1208 ... 53 ✓

2b) 

$A(x) = 2\pi x y$

$= 2\pi x \sin x$ ✓

Volume of shell $= 2\pi x \sin x \,\Delta x$

(ii) Total Volume $= 2\pi\int_0^{\pi/3} x \sin x \, dx$

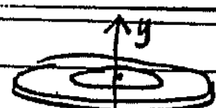
$u = x \quad v' = \sin x$

$u' = 1 \quad v = -\cos x$

$\therefore V = 2\pi \left\{ \left[-x\cos x\right]_0^{\pi/3} + \int_0^{\pi/3} \cos x \, dx \right\}$

$= 2\pi \left\{ -\frac{\pi}{6} - 0 + \left[\sin x\right]_0^{\pi/3} \right\}$

$= \frac{\pi}{3}(3\sqrt{3} - \pi)$ or equivalent ✓

3 a) (4 marks) 

$A(x) = \pi(x_2^2 - x_1^2)$

$= \pi(x_2 - x_1)(x_2 + x_1)$

Now $y^2 = (x-3)^2 = 4 - y^2$

$x - 3 = \pm\sqrt{4-y^2}$ ✓

$x_2 = 3 + \sqrt{4-y^2}$

$x_1 = 3 - \sqrt{4-y^2}$

$\therefore A(x) = \pi(2\sqrt{4-y^2})(6)$ ✓

$= 12\pi\sqrt{4-y^2}$

(ii) Total Vol $= \lim_{\Delta y \to 0} \sum_{-2}^{2} 12\pi\sqrt{4-y^2}\,\Delta y$

$V = 24\pi\int_0^2 \sqrt{4-y^2}\, dy$ ✓ for limits

$y = 2\sin\theta$
$dy = 2\cos\theta\, d\theta$

$= 24\pi\int_0^{\frac{\pi}{2}} 2\cos\theta . 2\cos\theta\, d\theta$ ✓ with change in limits

$= 48\pi\int_0^{\frac{\pi}{2}} \cos 2\theta + 1 \, d\theta$

$= 48\pi\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\pi/2}$

/5

$= 48\pi\left(0 + \frac{\pi}{2} - 0\right)$

$= \cancel{24}\pi^2$ ✓

Alternatively $V = 12\pi\int_{-2}^{2}\sqrt{4-y^2}\,dy$

$= 12\pi \times \frac{\pi 2^2}{2}$ Area of semicircle of radius 2

$= 24\pi^2$

must state this

Note: $\int_0^2 \sqrt{4-y^2}\, dy$ is area of Quadrant of circle, radius 2

different limits

2c) $\int_{-\pi/4}^{\pi/4} \frac{e^x}{1+e^x}\tan^2 x \, dx$

$= \int_0^{\pi/4}\left[\frac{e^x}{1+e^x} + \frac{e^{-x}}{1+e^{-x}}\right]\tan^2(-x)\, dx$ ✓

$= \int_0^{\pi/4}\left[\frac{e^x}{1+e^x} + \frac{1}{e^x+1}\right]\tan^2 x \, dx$ ✓

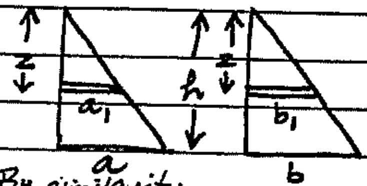
/4

$= \int_0^{\pi/4} \tan^2 x \, dx$

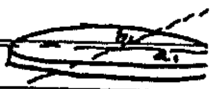
$= \int_0^{\pi/4} \sec^2 x - 1 \, dx$ ✓

$= \left[\tan x - x\right]_0^{\pi/4}$ ✓

$= 1 - \pi/4$

3b) 

Let a_1 and b_1 be semi axes of elliptical slice



By similarity,

$\frac{z}{h} = \frac{a_1}{a} \qquad \frac{b_1}{b_1} = \frac{z}{h}$

$\therefore a_1 = \frac{a}{h}$ ✓ $\quad b_1 = \frac{bz}{h}$ ✓

$\therefore$ Cross sectional area of slice $= \pi a_1 b_1$ ✓

$= \pi\frac{az}{h}.\frac{bz}{h} = \frac{\pi a b z}{h^2}$

(ii) Vol of Slice $= \frac{\pi a b z^2}{h^2}.\Delta z$

Total Vol $= \lim_{\Delta z \to 0}\sum_0^h \frac{\pi a b z^2}{h^2}$

$= \frac{\pi ab}{h^2}\int_0^h z^2\, dz$

$= \frac{\pi ab}{h^2}\left[\frac{z^3}{3}\right]_0^h$ ✓

/5

$= \frac{\pi ab}{h^2}.\frac{h^3}{3}$

$= \frac{\pi a b h}{3}$ ✓

c)(i) $I_0 = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \frac{\pi}{4}$ ✓

(ii) $I_n + I_{n+2} = \int_0^1 \frac{x^n}{1+x^2} + \frac{x^{n+2}}{1+x^2}\,dx$

$= \int_0^1 \frac{x^n(1+x^2)}{1+x^2}\,dx$ ✓

/4

$= \int_0^1 x^n\, dx$

$= \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$ as required ✓

(iii) $I_{n+2} = \frac{1}{n+1} - I_n$ Letting $n=6$ we get

$\therefore I_8 = \frac{1}{7} - I_6 = \frac{1}{7} - \left(\frac{1}{5} - I_4\right)$

$= \frac{1}{7} - \frac{1}{5} + \left(\frac{1}{3} - I_2\right)$ ✓

$= \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - \left(\frac{1}{1} - I_0\right) = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + \frac{\pi}{4}$