YEAR 12 EXTENSION 2 MATHEMATICS
JUNE 2007

NAME		RESULT	
DIRECTIONS	▪ Full working should be shown in every question. Marks may be deducted for careless or badly arranged work. ▪ Use black or blue pen only (*not pencils*) to write your solutions. ▪ No liquid paper is to be used. If a correction is to be made, one line is to be ruled through the incorrect answer.		

TIME ALLOWED: 70 MINUTES

1. Find $\displaystyle\int\frac{4x-6dx}{(x+1)(2x^2+3)}$ **3**

2. Find $\displaystyle\int(\ln x)^2\,dx$ by using the technique of integration by parts **3**

3. Find $\displaystyle\int\frac{x^2dx}{\sqrt{4-x^2}}$ **4**

4. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{5+4\cos x}$ by using the substitution $t=\tan\frac{x}{2}$. **4**

5. Find $\displaystyle\int\frac{dx}{\sqrt{7-2x-x^2}}$ **2**

6. i. Use the result $\displaystyle\int_0^a f(x)dx=\int_0^a f(a-x)dx$ to show that $\displaystyle\int_0^{\frac{\pi}{4}}\frac{\cot(x+\frac{\pi}{4})dx}{1-\sin 2x}=\frac{1}{2}\int_0^{\frac{\pi}{4}}\sec^2 x\tan x dx.$ **3**

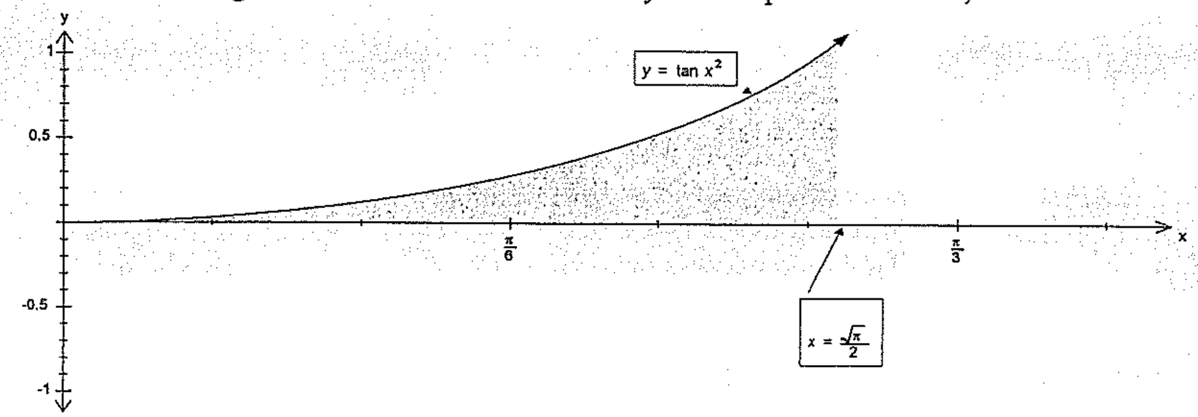
   ii. Hence show that $\displaystyle\int_0^{\frac{\pi}{4}}\frac{\cot(x+\frac{\pi}{4})dx}{1-\sin 2x}=\frac{1}{4}$ **2**

7. Use a suitable substitution to prove that $\displaystyle\int_{-1}^{0}x(1+x)^n\,dx=\frac{-1}{(n+1)(n+2)}$ **3**

8. i. Find $\frac{d}{dx}\left(\ln(\cos x^2)\right)$ **1**

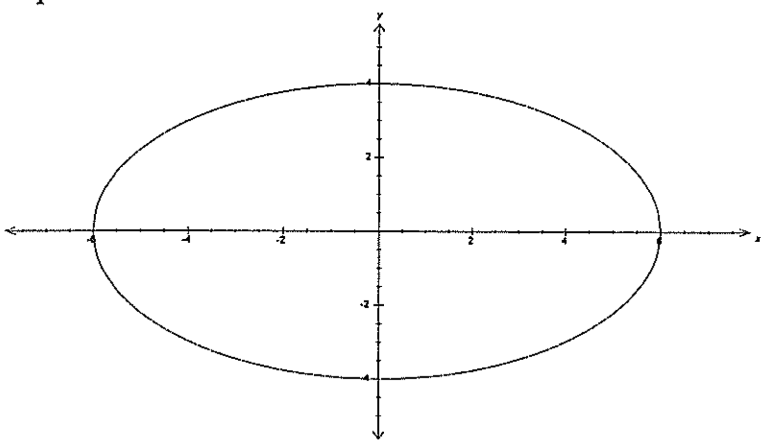
ii. The region in the plane bounded by the curve $y = \tan x^2$, the x axis and the line $x = \frac{\sqrt{\pi}}{2}$ **4**

shown in the diagram below is rotated about the y axis to produce a solid, S.



Use the method of cylindrical shells to show the volume of S is $\frac{\pi}{2}\ln 2$ units³.

9. A solid has an elliptical base with equation $\frac{x^2}{36} + \frac{y^2}{16} = 1$. Sections of the solid perpendicular to its base and parallel to the minor axis are semicircles.



i. Show that the area of a slice of the solid is given by $A(x) = \frac{2\pi}{9}(36 - x^2)$ units². **2**

ii. Hence, find the volume of the solid. **2**

10. The area between the parabola $y = 6x - x^2$ and the x axis is rotated about the y axis. When the area is rotated a line segment, L, at height y sweeps out an annulus.
The x ordinates of the end points of L are x_1 and x_2.

i. Show that the area of the annulus is equal to $A(y) = 12\pi\sqrt{9 - y}$ units². **2**

ii. Hence find the volume of the solid formed. **3**

11. $I_n = \int_0^1 x^n \sqrt{1-x}\, dx$. Show that for n>0, $I_n = \left(\frac{2n}{2n+3}\right) I_{n-1}$, where n is an integer. **4**

END OF EXAMINATION

EXTENSION 2 JUNE 2007

1 let $\int \frac{4x-6}{(x+1)(2x^2+3)}\,dx = \int \left(\frac{A}{x+1} + \frac{Bx+C}{2x^2+3}\right) dx$ (1) for either

$4x-6 = A(2x^2+3) + (Bx+C)(x+1)$

Put $x=-1$

$-10 = 5A + 0$

$A = -2$

Equating coefficients:

of x^2: $0 = 2A + B$

$B = 2 \times 2$

$B = 4$

of x^0: $-6 = 3A + C$

$C = 0$ $\therefore A=-2, B=4, C=0$ (1) or CFPA $A=-2$, $C=0$

[3]

$\therefore \int \frac{4x-6}{(x+1)(2x^2+3)}\,dx = \int \frac{-2}{x+1} + \frac{4x}{2x^2+3}\,dx$

$= -2\ln(x+1) + \ln(2x^2+3) + C$ (1) no CFPA if B omitted above

2. $\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln x\,dx$ $\quad u=(\ln x)^2 \quad v'=1$, $u' = \frac{2\ln x}{x} \quad v = x$ (1)

$= x(\ln x)^2 - 2\int 1.\ln x\,dx$ $\quad u = \ln x \quad v'=1$, $u' = \frac{1}{x} \quad v = x$ (1)

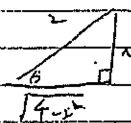
$= x(\ln x)^2 - 2\left(x\ln x - \int 1\,dx\right)$ [3]

$= x(\ln x)^2 - 2x\ln x + 2x + C$ (1)

3. $\int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin^2\theta . 2\cos\theta\,d\theta}{\sqrt{4(1-\sin^2\theta)}}$ (1) let $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$ (1)

$= \int \frac{4\sin^2\theta \cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$

$= \int 2(1-\cos 2\theta)\,d\theta$ [4]



$= 2\theta - \sin 2\theta + C$ (1)

$= 2\sin^{-1}\frac{x}{2} - 2\,\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C = 2\sin^{-1}\frac{x}{2} - \frac{x\sqrt{4-x^2}}{2} (+c)$ (1)

4. $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x} = \int_0^1 \frac{2dt}{(t^2+1)\left(5+4\frac{(1-t^2)}{(1+t^2)}\right)}$ (1)

let $t = \tan\frac{x}{2}$, $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$, $\frac{dt}{dx} = \frac{1}{2}(t^2+1)$, $\frac{dx}{dt} = \frac{2}{(t^2+1)}$

$= \int_0^1 \frac{2dt}{5t^2+5+4-4t^2}$

when $x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4}$, $t = 1$; when $x=0$, $t = \tan 0$, $t = 0$ (1)

$= \int_0^1 \frac{2dt}{t^2+9}$

$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^1$ (1)

[4]

$= \frac{2}{3}\left(\tan^{-1}\frac{1}{3} - \tan^{-1}0\right)$

$= \frac{2}{3}\tan^{-1}\frac{1}{3}$ (1)

5. $\int \frac{dx}{\sqrt{7-2x-x^2}} = \int \frac{dx}{\sqrt{8-(x+1)^2}}$ $\quad 7-2x-x^2 = 7-(x^2+2x+1)+1 = 8-(x+1)^2$ (1)

$= \sin^{-1}\left(\frac{x+1}{2\sqrt{2}}\right)(+C)$ (or equivalent expression $\sqrt{8}$) (1) [2]

6 a) $\int_0^{\frac{\pi}{4}} \frac{\cot(x+\frac{\pi}{4})\,dx}{1+\sin 2x} = \int_0^{\frac{\pi}{4}} \frac{\cot\left(\frac{\pi}{4}-x+\frac{\pi}{4}\right)dx}{1+\sin\left(2(\frac{\pi}{4}-x)\right)}$ (1)

$= \int_0^{\frac{\pi}{4}} \frac{\cot(\frac{\pi}{2}-x)}{1+\sin(\frac{\pi}{2}-2x)}$

$= \int_0^{\frac{\pi}{4}} \frac{\tan x\,dx}{1+\cos 2x}$ (1) [3]

$= \int_0^{\frac{\pi}{4}} \frac{\tan x\,dx}{2\cos^2 x} = \frac{1}{2}\int \tan x\sec^2 x\,dx$ (1)

b) $= \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan x \sec^2 x\,dx$ (1)

$= \frac{1}{2}\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{4}\left(\tan^2\frac{\pi}{4} - \tan 0\right) = \frac{1}{4}(1-0) = \frac{1}{4}$ (1) [2]

7. let $u = 1+x$ $du = dx$

when $x=0,\ u=1$

$x=-1,\ u=0$

$$\int_{-1}^{0} x(1+x)^n\,dx = \int_0^1 (u-1)u^n\,du$$ (1) including limits

$$= \int_0^1 u^{n+1} - u^n\,du$$

$$= \left[\frac{u^{n+2}}{n+2} - \frac{u^{n+1}}{n+1}\right]_0^1$$ (1)

$$= \frac{1}{n+2} - \frac{1}{n+1}$$

$$= \frac{n+1-(n+2)}{(n+1)(n+2)}$$ $\boxed{3}$

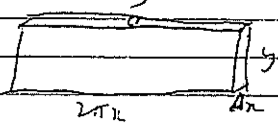
$$= \frac{-1}{(n+1)(n+2)}$$ (1)

$$\therefore \int_{-1}^{0} x(1+x)^n\,dx = \frac{-1}{(n+1)(n+2)}$$

8.(i) $$\frac{d}{dx}\left(\ln(\cos x^2)\right) = \frac{-2x\sin x^2}{\cos x^2}$$ $\boxed{1}$

$$= -2x\tan x^2$$ (1)

(ii) ~~Volume of~~ $= \int^{\sqrt{\pi}/2}$



Volume of shell, $\Delta V = 2\pi x y\,\Delta x$

$= 2\pi x\tan x^2\,\Delta x$

Volume of solid, $V = \lim_{\Delta x\to 0} \sum_{x=0}^{\sqrt{\pi}/2} 2\pi x\tan x^2\,\Delta x$

$$= \pi\int_0^{\sqrt{\pi}/2} 2x\tan x^2\,dx$$ (1)

$$= -\pi\left[\ln(\cos x^2)\right]_0^{\sqrt{\pi}/2}$$ (1)

$$= -\pi\left(\ln\left(\cos\frac{\pi}{4}\right) - \ln(\cos 0)\right)$$

$$= -\pi\left(\ln\frac{1}{\sqrt{2}} - \ln 1\right)$$ (1) $\boxed{4}$

$$= -\pi\left(\ln 2^{-1/2} - 0\right)$$

$\therefore V = \frac{\pi}{2}\ln 2$ units^3 as req'd. (1) ignore units

9. (i) Area of slice $= \frac{\pi y^2}{2}$ (1)

$$= \frac{\pi}{2}\left(16\left(1-\frac{x^2}{36}\right)\right)$$ $\boxed{2}$

$$= \frac{16\pi}{2}\left(\frac{36-x^2}{36}\right)$$

$$= \frac{2\pi}{9}\left(36-x^2\right)\text{ units}^2$$ (1) ignore units

(ii) Volume of slice $\Delta V = \frac{2\pi}{9}(36-x^2)\,\Delta x$

Volume $= \lim_{\Delta x\to 0}\sum_{x=-6}^{6} \frac{2\pi}{9}(36-x^2)\,\Delta x$

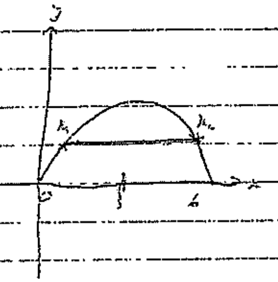
$$= \frac{4\pi}{9}\int_0^6 36 - x^2\,dx$$ (1)

$$= \frac{4\pi}{9}\left[36x - \frac{x^3}{3}\right]_0^6$$ $\boxed{2}$

$$= \frac{4\pi}{9}\left(216 - \frac{216}{3} - 0\right)$$

$= 64\pi$ units^3 ignore units (1)

10



(i) $A = \pi x_2^2 - \pi x_1^2$

$= \pi(x_2+x_1)(x_2-x_1)$

Now $x^2 - 6x + y = 0$

$$x = \frac{6\pm\sqrt{36-4y}}{2}$$

$$= \frac{6\pm 2\sqrt{9-y}}{2}$$

$$= 3\pm\sqrt{9-y}$$ (1)

$x_2 = 3 + \sqrt{9-y} \qquad x_1 = 3 - \sqrt{9-y}$

$\therefore A(y) = \pi\left(3+\sqrt{9-y} + 3 - \sqrt{9-y}\right)\left(3+\sqrt{9-y} - 3 + \sqrt{9-y}\right)$ $\boxed{2}$

$= \pi \times 6 \times 2\sqrt{9-y}$

$A(y) = 12\pi\sqrt{9-y}$ (1)

10 (ii) $\therefore$ Volume of slice, $\Delta V = 12\pi\sqrt{9-y}\,\Delta y$

Volume $= \lim\limits_{\Delta y \to 0} \sum\limits_{y=0}^{9} 12\pi\sqrt{9-y}\,\Delta y$ when $x=3$, $y=9$ (1)

$= 12\pi\int_0^9 (9-y)^{\frac{1}{2}}\,dy$

$= -12\pi \times \frac{2}{3}\left[(9-y)^{\frac{3}{2}}\right]_0^9$ (1) $\boxed{3}$

$= -8\pi\left((9-9)^{\frac{3}{2}} - (9-0)^{\frac{3}{2}}\right)$

$= -8\pi(0-27)$

$= 216\pi$ units^3. (1)

11. $I_n = \int_0^1 x^n\sqrt{1-x}\,dx$ $\quad u = x^n \quad v' = \sqrt{1-x}$; $\quad u' = nx^{n-1} \quad v = \frac{-2(1-x)^{\frac{3}{2}}}{3}$ (1)

$= \left[\frac{-2}{3}(1-x)^{\frac{3}{2}}x^n\right]_0^1 - \int_0^1 \frac{-2n}{3}x^{n-1}(1-x)^{\frac{3}{2}}\,dx$ (1)

$= (0-0) + \frac{2n}{3}\int_0^1 (1-x)\left(x^{n-1}\times\sqrt{1-x}\right)dx$ (1)

$= \frac{2n}{3}\left(\int_0^1 x^{n-1}\sqrt{1-x}\,dx - \int_0^1 x^n\sqrt{1-x}\,dx\right)$

$I_n + \frac{2n}{3}I_n = \frac{2n}{3}I_{n-1}$

$I_n\left(\frac{3+2n}{3}\right) = \frac{2n}{3}I_{n-1}$

$I_n = \frac{2n}{2n+3}I_{n-1}$ $\boxed{4}$ (1)

$\boxed{\text{TOTAL} = 45}$