QUESTION 1 (15 marks) Start on a new page

Marks

(a) i) Evaluate $\int_1^6 \frac{dx}{\sqrt{x+3}}$ 2

ii) Find $\int \frac{dx}{3+2\cos x}$ 3

iii) Find $\int \sqrt{\frac{2-x}{3+x}}\,dx$ 3

iv) Find $\int \sqrt{x}\log_e x\,dx.$ 3

b) Given that $\frac{4x-6}{(x+1)(2x^2+3)}$ can be written in the form

$$\frac{4x-6}{(x+1)(2x^2+3)} \equiv \frac{a}{x+1} + \frac{bx+c}{2x^2+3}$$ where a, b and c are real numbers

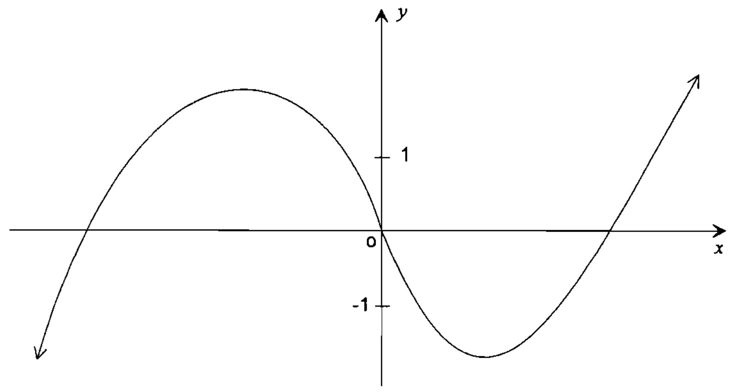
i) Find a, b and c. 2

ii) Hence find $\int \frac{4x-6\,dx}{(x+1)(2x^2+3)}$ 2

QUESTION 2 (15 marks) **Start on a new page**

Marks

(a) The diagram shows the graph of $y = f(x)$



Draw separate sketches (at least $\frac{1}{3}$ of a page) of the graphs of the following

i) $y = \dfrac{1}{f(x)}$ 2

ii) $y = f(|x|)$ 2

iii) $y = \log_e f(x)$ 2

iv) $y^2 = f(x)$ 2

(b) Let $f(x) = \dfrac{x-2}{(x+1)(5-x)}$ 3

Draw a neat sketch of $y = f(x)$ showing all asymptotes and intercepts with the coordinate axes. You are not required to use calculus.

(c) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx.$ prove that

$I_n = \dfrac{n-1}{n} I_{n-2}$ and hence find 3

$\int_0^{\frac{\pi}{2}} \cos^5 x \, dx.$ 1

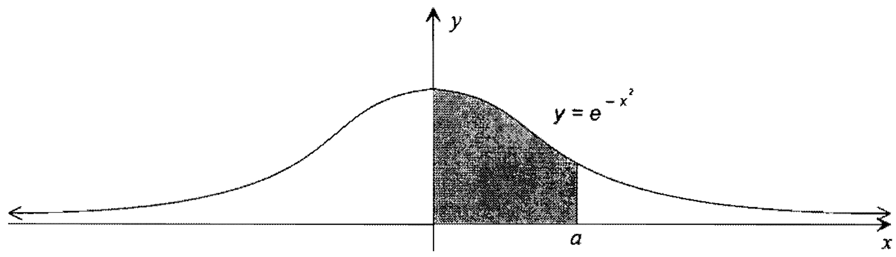
QUESTION 3 (15 marks) Start on a new page

Marks

(a) The region bounded by the curve $y = e^{-x^2}$, the coordinate axes and the line $x = a\,(a > 0)$ is rotated about the y axis to form a solid of revolution.

i) Use the method of cylindrical shells to find the volume of the solid. 4

ii) What is the limiting value of the volume as $a \to \infty$. 1



(b) The base of a solid is the region in the xy plane enclosed by the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Each cross section perpendicular to the x axis is a semi ellipse whose major axis is in the base and whose minor axis is half the major axis.

i) Find the volume V_1 of the solid given that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $A = \pi ab$ 4

ii) If the base of this solid was rotated about the x axis to form a solid of volume V_2 find the ratio $V_1 : V_2$ 1

(c) Use result that

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ to show that

$$\int_0^{\frac{\pi}{4}} \frac{\operatorname{cosec}2x - 1}{\operatorname{cosec}2x + 1}\,dx = \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{1 + \cos 2x}\,dx$$ 3

Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{\operatorname{cosec}2x - 1}{\operatorname{cosec}2x + 1}\,dx$ 2

END OF EXAMINATION

EXT 2 JUNE ASSESSMENT 2009

1 a i) $\int_1^6 \frac{dx}{\sqrt{x+3}}$

$= \left[2\sqrt{x+3}\right]_1^6$ 1   (2)

$= 2\sqrt{9} - 2\sqrt{4}$

$= 2$ 1

ii) $\int \frac{dx}{3+2\cos x}$

let $t = \tan\frac{x}{2}$

$\therefore I = \int \frac{1}{3+\frac{2(1-t^2)}{1+t^2}} \cdot \frac{2dt}{1+t^2}$ 1

$= \int \frac{2dt}{5+t^2}$ (3)

$= \frac{2}{\sqrt5}\tan^{-1}\frac{t}{\sqrt5}$ 1

$= \frac{2}{\sqrt5}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt5}\right) + C$ 1

iii) $\int \sqrt{\frac{2-x}{3+x}}\,dx$

$= \int \frac{2-x}{\sqrt{(3+x)(2-x)}}\,dx$ 1

$= \int \frac{2-x}{\sqrt{6-x-x^2}}\,dx$

$= \int \frac{\frac12(-1-x)+\frac52}{\sqrt{6-x-x^2}}\,dx$

$= \sqrt{6-x-x^2} + \frac52\int \frac{dx}{\sqrt{6-(x^2+x+\frac14)+\frac14}}$

$= \sqrt{6-x-x^2} + \frac52\int \frac{dx}{\sqrt{\frac{25}{4}-(x+\frac12)^2}}$ 1   (3)

$= \sqrt{6-x-x^2} + \frac52\sin^{-1}\left(\frac{x+\frac12}{\frac52}\right)$

$= \sqrt{6-x-x^2} + \frac52\sin^{-1}\left(\frac{2x+1}{5}\right) + C$ 1

iv) $\int \sqrt{x}\ln x\,dx$

let $u = \ln x$ $v' = x^{\frac12}$

$u' = \frac1x$ $v' = \frac23 x^{\frac32}$ 1

$\therefore I = \frac23 x\sqrt{x}\ln x - \frac23\int x^{\frac12}\,dx$ 1

$= \frac23 x\sqrt{x}\ln x - \frac49 x\sqrt{x} + C$

b) $\frac{4x-6}{(x+1)(2x^2+3)} \equiv \frac{a}{x+1} + \frac{bx+c}{2x^2+3}$

$a(2x^2+3) + (bx+c)(x+1) \equiv 4x-6$

SUBST $x=-1 \Rightarrow 5a=-10 \therefore a=-2$ *

COEF OF $x^2 \Rightarrow 2a+b=0$

$\therefore -4+b=0 \therefore b=4$ *

CONST TERM $\Rightarrow 3a+c=-6$ (2)

$-6+c=-6 \therefore c=0$

(-1 MISTAKE) 2

$\therefore I = \int \frac{4x\,dx}{2x^2+3} - \int \frac{2}{x+1}\,dx$ (2)

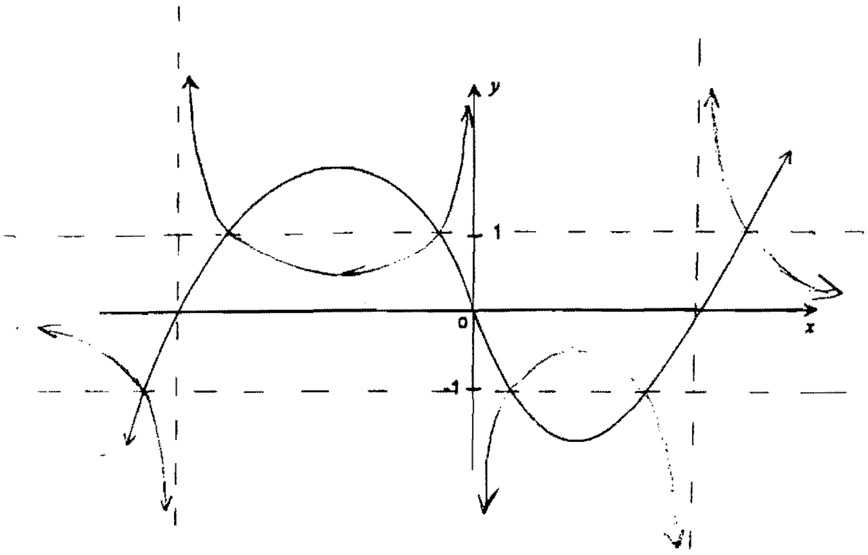
$= \ln(2x^2+3) - 2\ln(x+1) + C$ 1 1

OR

$= \ln\left(\frac{2x^2+3}{(x+1)^2}\right) + c$

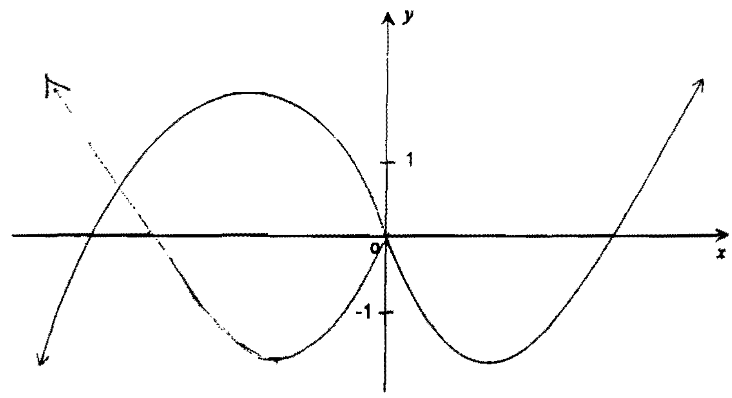
2 a)

i)



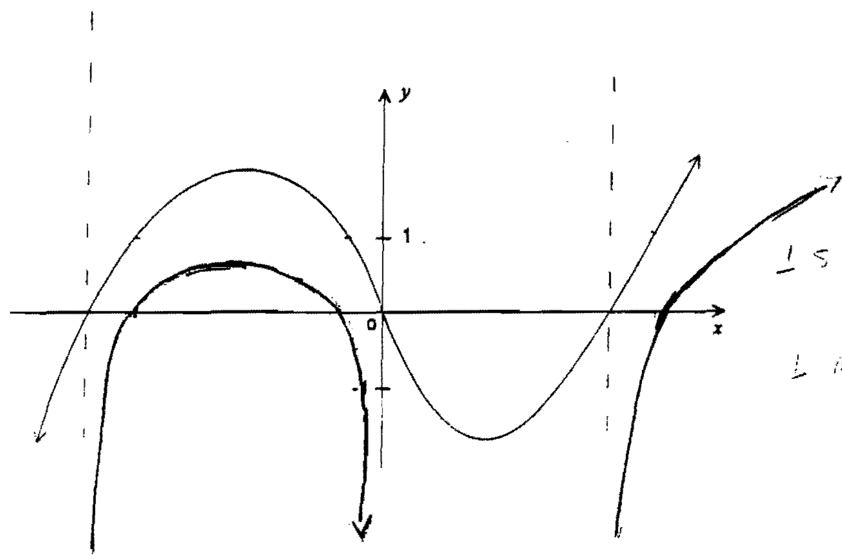
1 GENERAL SHAPE
1 ASYMPTOTES AND DISCONTINUITIES
(2)

ii)



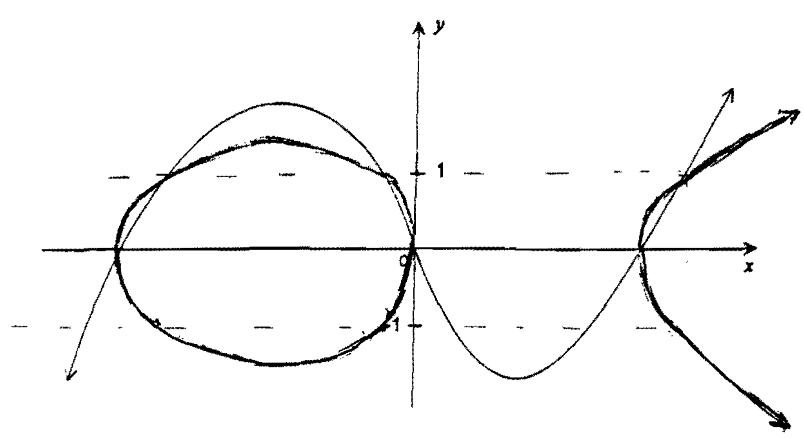
1 LEFT Y AXIS (2)
1 RIGHT Y AXIS

iii)



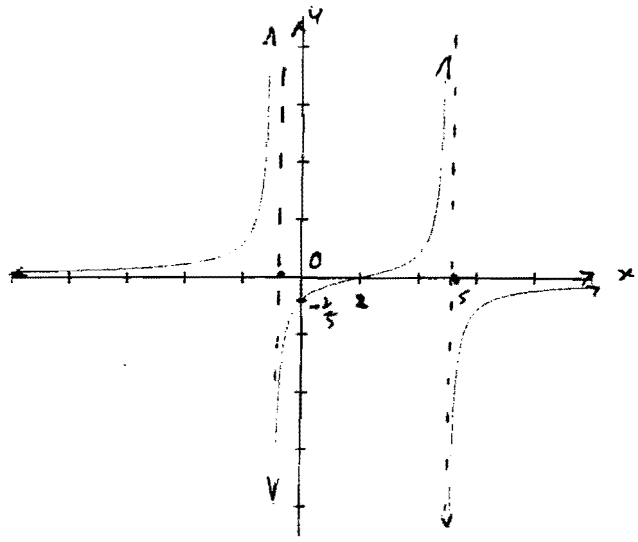
1 SHAPE INCLUDING [illegible]
1 INTERCEPTS [illegible]

iv)



1 SHAPE W.R.T. [illegible]
1 VERTICAL [illegible] [illegible]
PTS WHERE $y = \mp 1$

b)



1 GENERAL SHAPE

1 ASYMPTOTES

1 X, Y INTERCEPTS

(3)

c) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$

let $u = \cos^{n-1} x \quad v' = \cos x$

$u' = (n-1)\cos^{n-2} x \sin x \quad V = \sin x$ 1

$\therefore I_n = \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} (\cos^{n-2} x - \cos^n x) \, dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x \, dx$ 1

$I_n = (n-1) I_{n-2} - (n-1) I_n$

(3)

$(n-1+1) I_n = (n-1) I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$ 1

$\int_0^{\frac{\pi}{2}} \cos^5 x \, dx = I_5$

$I_5 = \frac{4}{5} I_3$

$I_3 = \frac{2}{3} I_1$

$I_1 = \int_0^{\frac{\pi}{2}} \cos x \, dx$

$= \left[\sin x\right]_0^{\frac{\pi}{2}}$

$= 1$ (1)

$\therefore I_5 = \frac{4}{5} \cdot \frac{2}{3} \cdot 1$

$= \frac{8}{15}$ 1

$\boxed{15}$

③ a) i) $V = 2\pi \int_0^a xy\,dy$ 1

$= 2\pi \int_0^a x e^{-x^2} dx$ 1

$= 2\pi \left[-\frac{1}{2} e^{-x^2} \right]_0^a$

$= 2\pi \left[-\frac{1}{2} e^{-a^2} + \frac{1}{2} \right]$ 1

$= \pi \left[1 - e^{-a^2} \right]$ cu unit 1

ii) as $a \to \infty$ ⑤

$e^{-a^2} \to 0$

∴ LIMITING VALUE OF

$V = \pi$ cu unit 1

b) i)

$V = 2 \int_0^4 \frac{\pi y^2}{4} dx$ 1

$= \frac{\pi}{2} \int_0^4 y^2 dx$

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

$\therefore y^2 = 9\left(1 - \frac{x^2}{16}\right)$

$\therefore V = \frac{9\pi}{2} \int_0^4 \left(1 - \frac{x^2}{16}\right) dx$ 1

$= \frac{9\pi}{2} \left[x - \frac{x^3}{48} \right]_0^4$ 1

$= \frac{9\pi}{2} \left[4 - \frac{4}{3} \right]$

$= 12\pi$ cu unit 1

ii) for V_2 $\delta A = \pi \cdot 4y^2$

$\therefore V_2 = 4V_1$ ⑤

$\therefore V_1 : V_2 = 1 : 4$ 1

c) $I = \int_0^{\frac{\pi}{4}} \frac{\operatorname{cosec} 2(\frac{\pi}{4} - x) - 1}{\operatorname{cosec} 2(\frac{\pi}{4} - x) + 1} dx$

$= \int_0^{\frac{\pi}{4}} \frac{\operatorname{cosec}(\frac{\pi}{2} - 2x) - 1}{\operatorname{cosec}(\frac{\pi}{2} - 2x) + 1} dx$ 1

$= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{\cos 2x} - 1}{\frac{1}{\cos 2x} + 1} dx$ 1

$= \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{1 + \cos 2x} dx$

$= \int_0^{\frac{\pi}{4}} \frac{1 - (1 - 2\sin^2 x)}{1 + (2\cos^2 x - 1)} dx$

$= \int_0^{\frac{\pi}{4}} \tan^2 x\, dx$ 1

$= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\, dx$ ⑤

$= \left[\tan x - x \right]_0^{\frac{\pi}{4}}$ 1

$= 1 - \frac{\pi}{4}$ 1

15