BAULKHAM HILLS HIGH SCHOOL
MATHEMATICS

Year 12 Extension 2 Task 3 — Thursday 27th May 2010

Instructions:
a) Write all your answers on your own paper.
b) Show all necessary working.
c) Marks may be deducted for careless or badly arranged work.

Time Allowed: 70 minutes

Question 1 ***(15 marks)*** — ***Marks***

Find the following integrals:

(i) $\int \sin^4 x\, dx$ — *3*

(ii) $\int \frac{dx}{x^2\sqrt{1+x^2}}$ — *3*

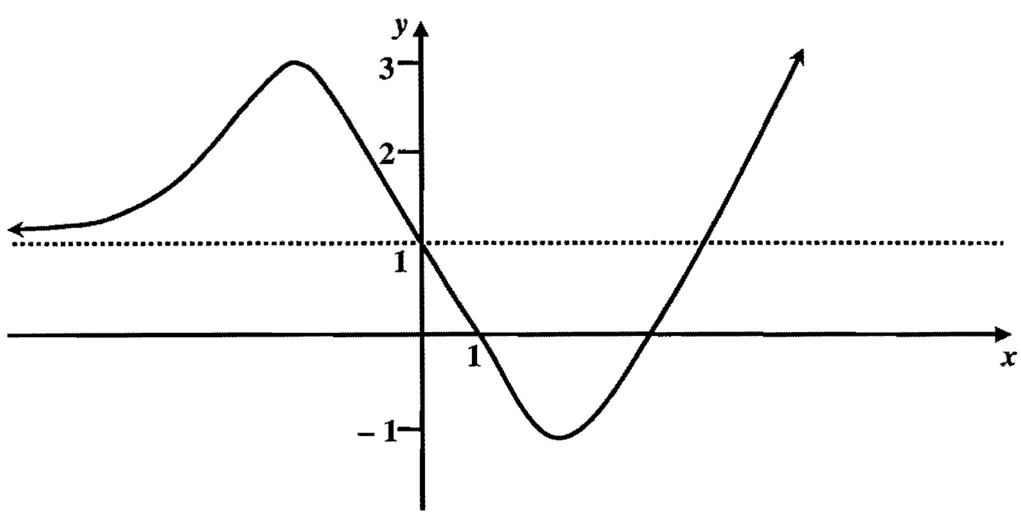
(iii) $\int x \tan^{-1} x\, dx$ — *3*

(iv) $\int \frac{x-1}{\sqrt{x^2+2x+3}}\, dx$ — *3*

(v) $\int \frac{5-2x}{(x-1)^2(x+2)}\, dx$ — *3*

Question 2 ***(12 marks)*** Use a *separate* piece of paper

a) The diagram shows the graph $y = f(x)$



Draw separate one-third page sketches of the graphs of the following;

(i) $y = f(1-x)$ — 2

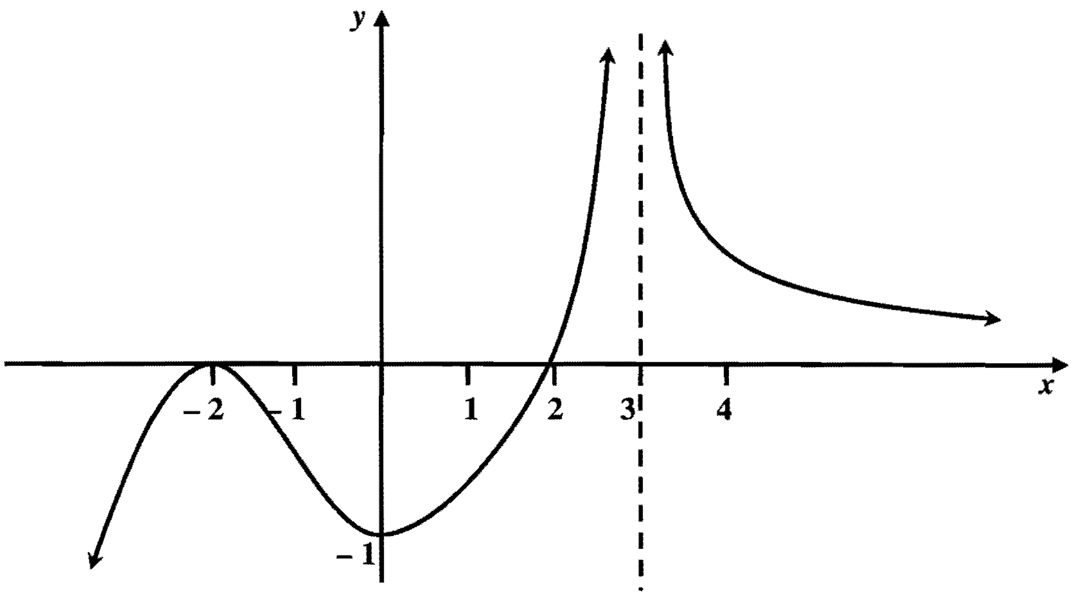
(ii) $y = \frac{1}{f(x)}$ — 2

(iii) $y = \left[f(x)\right]^2$ — 2

(iv) $y = \tan^{-1} f(x)$ — 2

Question 2...*continued* *Marks*

b) Given the graph of $y = f'(x)$ below, sketch the graph of $y = f(x)$ 4



Question 3 ***(13 marks)*** Use a *separate* piece of paper

a) $P\left(cp, \frac{c}{p}\right)$, $Q\left(cq, \frac{c}{q}\right)$ and $R\left(cr, \frac{c}{r}\right)$ are three points on the same branch of the hyperbola $xy = c^2$

(i) Show the equation of the tangent at P is $x + p^2 y - 2cp = 0$ 2

(ii) The tangents at P and Q intersect U. Find the coordinates of U 2

(iii) The origin, U and R are collinear, find the relationship between p, q and r. 2

b) (i) Prove that $\tan\frac{x}{2} \equiv \operatorname{cosec} x - \cot x$ 1

(ii) Show that $\int \frac{dx}{1+\cos x} = \operatorname{cosec} x - \cot x + c$ 3

(iii) Hence, or otherwise, show that $\int \frac{x + \sin x}{1+\cos x} dx = x \tan\frac{x}{2} + c$ 3

Question 4 ***(14 marks)*** Use a *separate* piece of paper

a) $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ are points on the rectangular hyperbola $xy = 9$

(i) Find the equation of the chord PQ. *2*

(ii) Find the coordinates of N, the midpoint of PQ. *2*

(iii) If the chord PQ is a tangent to the parabola $y^2 = 3x$, prove that the locus of N is $3x = -8y^2$ *3*

b) Let $I_n = \int_1^e (\ln x)^n \, dx$, where n is a positive integer.

(i) Show that $I_n = e - nI_{n-1}$ *2*

(ii) Let $J_n = \frac{I_n}{n!}$. Show that $\frac{1}{e}(1 + J_{10}) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + \frac{1}{10!}$ *3*

(iii) It can be shown that $\sum_{r=2}^{n} \frac{(-1)^r}{r!} = \frac{1}{e}\left[1 + (-1)^n J_n\right]$ for all positive integers n. *2*

Deduce the sum to infinity of the series $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots$, justifying your answer carefully.

END OF EXAMINATION

Extension 2 Task 3 2010 Solutions

Question 1 (15)

(i) $\int \sin^4 x \, dx$

$= \frac{1}{4}\int (1-\cos 2x)^2 dx$ ✓

$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) dx$

$= \frac{1}{4}\int (1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)) dx$

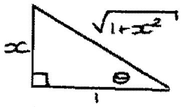
$= \frac{1}{4}\int (\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x) dx$ ✓

$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$ ✓ (3)

(ii) $\int \frac{dx}{x^2\sqrt{1+x^2}}$ $x = \tan\theta$, $dx = \sec^2\theta \, d\theta$ ✓

$= \int \frac{\sec^2\theta \, d\theta}{\tan^2\theta \sec\theta}$

$= \int \frac{\sec\theta \, d\theta}{\tan^2\theta}$



$= \int \frac{1}{\cos\theta} \times \frac{\cos^2\theta}{\sin^2\theta} d\theta$

$= \int \frac{\cos\theta}{\sin^2\theta} d\theta$ ✓ $u = \sin\theta$, $du = \cos\theta \, d\theta$

$= \int \frac{du}{u^2}$

$= -\frac{1}{u} + c$

$= -\text{cosec}\,\theta + c$

$= -\frac{\sqrt{1+x^2}}{x} + c$ ✓ (3)

(iii) $\int x \tan^{-1}x \, dx$ $u = \tan^{-1}x$, $v = \frac{1}{2}x^2$, $du = \frac{dx}{1+x^2}$, $dv = x\,dx$

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2} dx$ ✓

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \left[1 - \frac{1}{1+x^2}\right] dx$ ✓

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + c$ ✓ (3)

(iv) $\int \frac{x-1}{\sqrt{x^2+2x+3}} dx$

$= \frac{1}{2}\int \frac{(2x+2)\,dx}{\sqrt{x^2+2x+3}} - 2\int \frac{dx}{\sqrt{(x+1)^2+2}}$ ✓ ✓

$= \sqrt{x^2+2x+3} - 2\log(x+1+\sqrt{x^2+2x+3}) + c$ ✓ (3)

(v) $\int \frac{5-2x}{(x-1)^2(x+2)} dx$

$\frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)} = \frac{5-2x}{(x-1)^2(x+2)}$

$A(x-1)(x+2) + B(x+2) + C(x-1)^2 = 5-2x$

$x=1$: $3B=3$, $B=1$ $x=-2$: $9C=9$, $C=1$ $x=2$: $4A+4B+C=1$, $4A+5=1$, $4A=-4$, $A=-1$ ✓

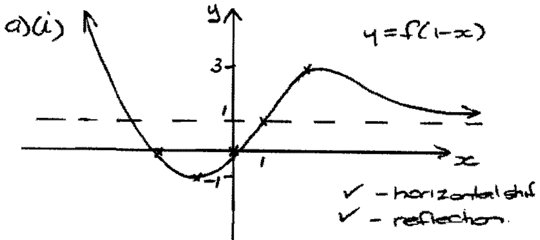
$\int \frac{5-2x}{(x-1)^2(x+2)} dx = \int \left[\frac{-1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x+2}\right] dx$ ✓

$= -\log(x-1) - \frac{1}{x-1} + \log(x+2)$ ✓

$= \log\left(\frac{x+2}{x-1}\right) - \frac{1}{x-1}$ (3)

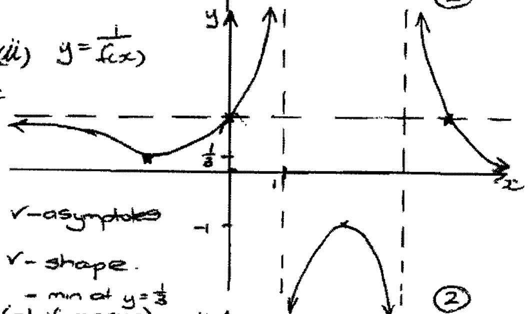
Question 2 (12)

a)(i)



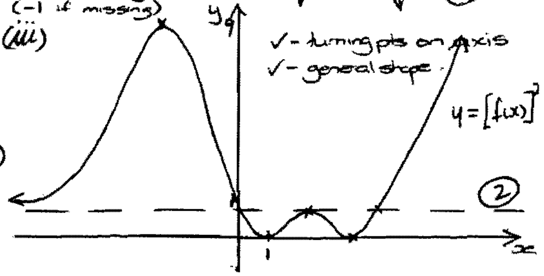
✓ - horizontal shift
✓ - reflection (2)

(ii) $y = \frac{1}{f(x)}$



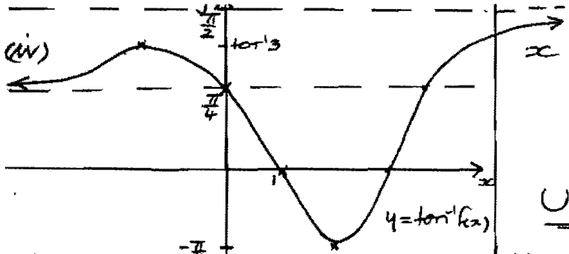
✓ - asymptotes
✓ - shape.
- min at $y=\frac{1}{3}$ (-1 if missing) (2)

(iii)



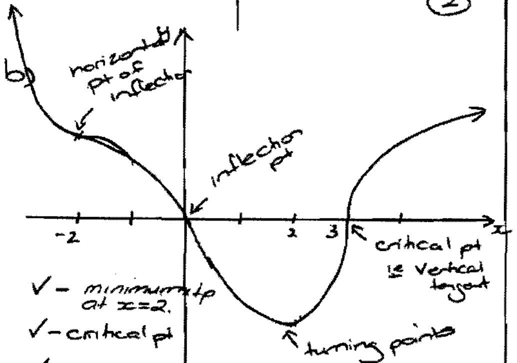
✓ - turning pts on axis
✓ - general shape (2)

(iv)



✓ - asymptotes
✓ - general shape (2)

b)



✓ - minimum tp at $x=2$.
✓ - critical pt
✓ pts of inflection and turning pt
✓ general shape (4)

Question 3 (13)

a) $xy = c^2$

$y = \frac{c^2}{x}$

$\frac{dy}{dx} = \frac{-c^2}{x^2}$

at $x = cp$, $\frac{dy}{dx} = \frac{-c^2}{c^2p^2} = -\frac{1}{p^2}$ ✓

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$ ✓

$x + p^2y - 2cp = 0$ (2)

(ii) tangent at Q: $x + q^2y - 2cq = 0$

$x + p^2y = 2cp$

$x + q^2y = 2cq$

$(p^2-q^2)y = 2c(p-q)$

$y = \frac{2c(p-q)}{(p+q)(p-q)}$

$y = \frac{2c}{p+q}$ ✓

$x + \frac{2cp^2}{p+q} = 2cp$

$x = \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$

$x = \frac{2cpq}{p+q}$ ✓

U is $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$ (2)

(iii) $m_{OU} = m_{OR}$ ✓ (or equiv)

$\frac{\frac{2c}{p+q}}{\frac{2cpq}{p+q}} = \frac{\frac{c}{r}}{cr}$

$\frac{1}{pq} = \frac{1}{r^2}$

$pq = r^2$ ✓ (2)

b)(i) $\text{cosec}\,x - \cot x$

$= \frac{1+t^2}{2t} - \frac{1-t^2}{2t}$

$= \frac{2t^2}{2t}$

$= t$

$= \tan\frac{x}{2}$ ✓ (1)

(ii) $\int \frac{dx}{1+\cos x}$ $t = \tan\frac{x}{2}$, $dx = \frac{2dt}{1+t^2}$

$= \int \frac{\frac{2dt}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}}$ ✓

$= \int \frac{2\,dt}{2}$

$= \int dt$ ✓

$= t + c$

$= \tan\frac{x}{2} + c$

$= \text{cosec}\,x - \cot x + c$ ✓ (3)

OR $\int \frac{dx}{1+\cos x} = \int \frac{1-\cos x}{1-\cos^2 x} dx$

$= \int \frac{1-\cos x}{\sin^2 x} dx$

$= \int [\text{cosec}^2 x - \cot x \,\text{cosec}\, x]$

$= -\cot x + \text{cosec}\,x + c$

(iii) $\int \frac{x+\sin x}{1+\cos x}\,dx$

$= \int \frac{x\,dx}{1+\cos x} + \int \frac{\sin x\,dx}{1+\cos x}$ ✓

$u = x \qquad v = \operatorname{cosec} x - \cot x$

$du = dx \qquad dv = \frac{dx}{1+\cos x}$

$= x(\operatorname{cosec} x - \cot x) - \int(\operatorname{cosec} x - \cot x)dx - \log(1+\cos x) + c$ ✓

$= x\tan\frac{x}{2} + \log(\operatorname{cosec} x + \cot x) + \log(\sin x) + \log(1+\cos x) + c$

$= x\tan\frac{x}{2} + \log\left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right) + \log(\sin x) - \log(1+\cos x) + c$

$= x\tan\frac{x}{2} + \log(1+\cos x) - \log(1+\cos x) + c$ ✓

$= \underline{\underline{x\tan\frac{x}{2}}}$ (3)

Question 4 (14)

a)(i) $m_{PQ} = \dfrac{\frac{3}{p} - \frac{3}{q}}{3p-3q}$

$= \dfrac{\frac{q-p}{pq}}{p-q}$

$= -\frac{1}{pq}$ ✓

$y - \frac{3}{p} = -\frac{1}{pq}(x - 3p)$

$pqy - 3q = -x + 3p$

$\underline{\underline{x + pqy - 3(p+q) = 0}}$ ✓ (2)

(ii) $N\left(\frac{3p+3q}{2}, \frac{\frac{3}{p}+\frac{3}{q}}{2}\right)$ ✓

$= \underline{\underline{\left(\frac{3p+3q}{2}, \frac{3p+3q}{2pq}\right)}}$ ✓ (2)

(iii) $y^2 = 3x$

$x = \frac{1}{3}y^2$

$\frac{1}{3}y^2 + pqy - 3(p+q) = 0$

$y^2 + 3pqy - 9(p+q) = 0$

If tangent then $\Delta = 0$ ✓

ie $9p^2q^2 + 36(p+q) = 0$

$p^2q^2 = -4(p+q)$ ✓

$3x = \frac{9(p+q)}{2} \qquad -8y^2 = -8\left(\frac{3(p+q)}{2pq}\right)^2$

$= \frac{18(p+q)^2}{p^2q^2}$

$= \frac{-18(p+q)^2}{-4(p+q)}$ ✓

$= \frac{9(p+q)}{2}$

$\therefore$ $\underline{\text{locus of } N \text{ is } 3x = -8y^2}$ (3)

b) $I_n = \int_1^e (\log x)^n dx$

$u = (\log x)^n \qquad v = x$

$du = \frac{n(\log x)^{n-1}}{x}dx \qquad dv = dx$ ✓

$I_n = \left[x(\log x)^n\right]_1^e - n\int_1^e (\log x)^{n-1}dx$ ✓

$= \underline{\underline{e - nI_{n-1}}}$ (2)

(ii) $\frac{1}{e}(1 + J_{10})$

$= \frac{1}{e}\left(1 + \frac{I_{10}}{10!}\right)$

$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{10I_9}{10!}\right)$ ✓

$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{I_9}{9!}\right)$

$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{e}{9!} + \frac{9I_8}{9!}\right)$ ✓

$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{e}{9!} + \frac{I_8}{8!}\right)$

$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{e}{9!} + \frac{e}{8!} - \dots + \frac{e}{2!} - \frac{e}{1!} + \frac{I_0}{0!}\right)$

$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{e}{9!} + \frac{e}{8!} - \dots + \frac{e}{2!} - \frac{e}{1!} + \int_1^e dx\right)$

$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{e}{9!} + \frac{e}{8!} - \dots + \frac{e}{2!} - e + [x]_1^e\right)$

$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{e}{9!} + \frac{e}{8!} - \dots + \frac{e}{2!} - e + e - 1\right)$

$= \frac{1}{e}\left(\frac{e}{10!} - \frac{e}{9!} + \frac{e}{8!} - \frac{e}{7!} + \frac{e}{6!} - \frac{e}{5!} + \frac{e}{4!} - \frac{e}{3!} + \frac{e}{2!}\right)$

$= \underline{\underline{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + \frac{1}{10!}}}$ ✓ (3)

(iii) $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots$

$= \sum_{r=2}^{\infty} \frac{(-1)^r}{r!}$

$= \lim_{n\to\infty} \frac{1}{e}\left[1 + (-1)^n J_n\right]$

$= \lim_{n\to\infty} \frac{1}{e}\left[1 + (-1)^n \frac{I_n}{n!}\right]$

as $n \to \infty$, $n! \to \infty$

for $n > 0$, $0 < I_n < e$

$\therefore \lim_{n\to\infty} \frac{I_n}{n!} = 0$

Thus

$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots = \underline{\underline{\frac{1}{e}}}$ (2)

✓ answer

✓ justification