BAULKHAM HILLS HIGH SCHOOL
MARKING COVER SHEET

2011
YEAR 12 JUNE ASSESSMENT TASK
EXTENSION 2

STUDENT NUMBER:___________________

TEACHERS NAME:___________________

QUESTION	MARK
1	
2	
3	
4	
TOTAL	/
PERCENTAGE	%

Topics Tested: Integration and Volumes

YEAR 12 EXTENSION 2 MATHEMATICS ASSESSMENT JUNE 2011

TIME : 70 MINUTES

NAME		RESULT	

DIRECTIONS
- Full working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Use black or blue pen only (*not pencils*) to write your solutions.
- No liquid paper or correction tape is to be used. If a correction is to be made, one line is to be ruled through the incorrect answer.

Question 1 (10 marks) — **Marks**

a) Find the indefinite integrals:

i) $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}}$ — 2

ii) $\displaystyle\int \frac{dx}{x^2+2x+5}$ — 3

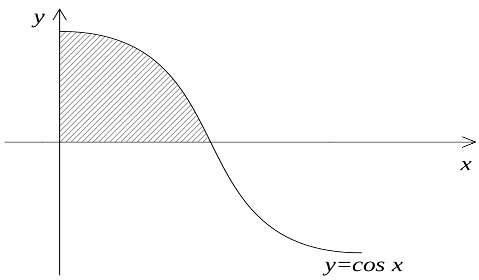
iii) $\int \sin^{-1} x \ dx$ — 2

b) Evaluate: — 3

$$\int_0^2 \frac{x^2}{x^6+64}\ dx$$

Question 2 (11 marks) - Start a new page

a) Use the method of cylindrical shells to find the volume of the solid generated when shaded region is rotated about the line $x = \pi$. — 4

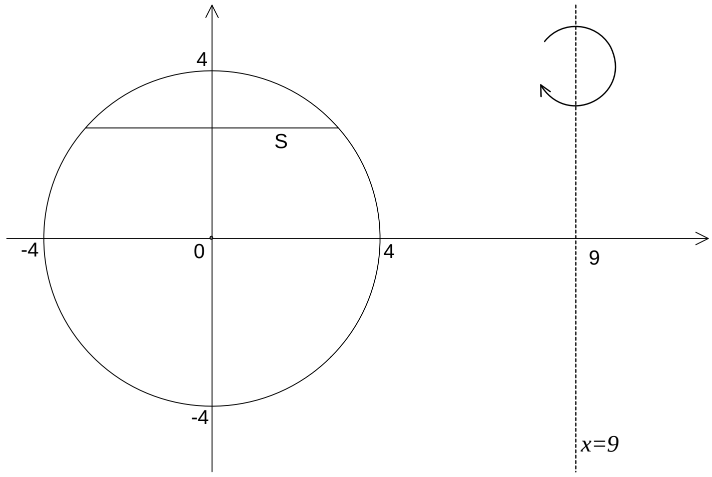


b) i) Show that $\displaystyle\int_0^1 \frac{1}{(5x+3)(x+1)}\ dx = \frac{1}{2}\ln\frac{4}{3}$ — 3

ii) Hence find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{4\sin x - \cos x + 4}\ dx$ using the substitution $t = \tan\frac{x}{2}$ — 4

Question 3 (12 marks) - Start a new page **Marks**

a) The circle $x^2 + y^2 = 16$ is rotated about the line $x = 9$ to form a ring (i.e. a torus). When the circle is rotated, the line segment S at height y sweeps out an annulus.



i) Show that the area of the annulus is equal to $36\pi\sqrt{16 - y^2}$. **3**

ii) Hence find the volume of the ring. **3**

b) Given $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x \; dx$,

i) Show that $I_n + I_{n-2} = \dfrac{1}{n-1}$ where n is an integer and $n \geq 3$ **4**

ii) Hence evaluate I_7. **2**

Question 4 (11 marks) - Start a new page **Marks**

a) A certain solid has a circular base of radius 4 units. The centre of the base is the origin. The cross-sections at right-angles to the x-axis are isosceles triangles. The height h of each of these triangles is given by $h = 16 - x^2$.

i) Show that the volume of the solid is given by $V = 2\displaystyle\int_0^4 (16 - x^2)^{\frac{3}{2}} \; dx$. **2**

ii) Hence find the volume V **4**

b) i) Show that $\int_0^{2a} f(x)dx = \int_0^a [f(x) + f(2a - x)]dx$ **3**

ii) Hence find the value of the constant k given that $\int_0^{\pi} \frac{x}{2+sinx} dx = k \int_0^{\pi/2} \frac{1}{2+sinx} dx$ **2**

-- End of Exam --

Ext 2 June Task 2011 SOLUTIONS.

Question 1.

a) i) $\int \frac{dx}{\sqrt{4-9x^2}}$ $\quad \left[\begin{array}{l} u=3x \\ du=3.dx \end{array} \right.$

$= \int \frac{\frac{1}{3}.du}{\sqrt{4-u^2}}$

$= \frac{1}{3} \sin^{-1} \frac{u}{2} + c$

$= \frac{1}{3} \sin^{-1} \frac{3x}{2} + c$ $\quad$ $\sin^{-1} = 1$ The rest $= 1$

ii) $\int \frac{dx}{x^2+2x+5}$

$= \int \frac{dx}{(x+1)^2+4}$ $\quad 1$

$= \frac{1}{2} \tan^{-1} \frac{x+1}{2} + c$ $\quad \to 1$ $\quad 1$

iii) $\int \sin^{-1} x .1.dx$

$= uv - \int vu'.dx$ $\quad \left[\begin{array}{ll} u=\sin^{-1}x & u'=\frac{1}{\sqrt{1-x^2}} \\ v=x & v'=1 \end{array} \right.$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}.dx$

$= x\sin^{-1}x - \frac{1}{2}\int \frac{2x}{\sqrt{1-x^2}}.dx$

$= x\sin^{-1}x + \frac{1}{2}\int \frac{du}{\sqrt{u}}$ $\quad \left[\begin{array}{l} u=1-x^2 \\ du=-2x.dx \end{array} \right.$

$= x\sin^{-1}x + \frac{1}{2}.2u^{1/2} + c$

$= \underbrace{x\sin^{-1}x}_{1} + \underbrace{\sqrt{1-x^2}}_{1} + c.$

b) $\int_0^2 \frac{x^2}{x^6+64}.dx$ $\quad \left[\begin{array}{l} u=x^3 \\ du=3x^2.dx \\ \text{If } x=0,\ u=0 \\ \text{If } x=2,\ u=8 \end{array} \right.$

$= \int_0^8 \frac{\frac{1}{3}.du}{u^2+64}$ $\quad 1$

$= \frac{1}{3}\left[\frac{1}{8}\tan^{-1}\frac{u}{8}\right]_0^8$ $\quad 1$

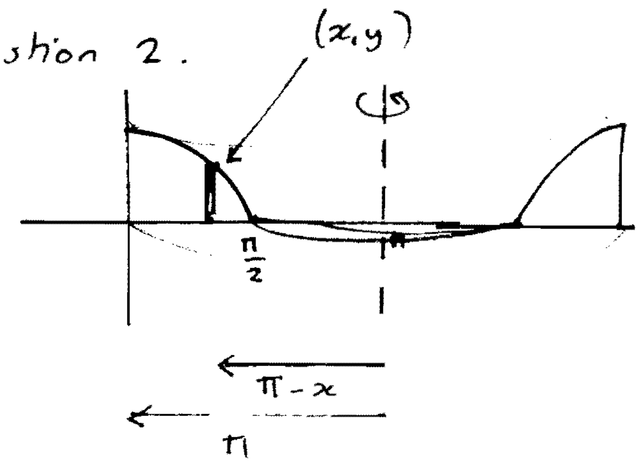
$= \frac{1}{24}\left(\tan^{-1}1 - \tan^{-1}0\right)$

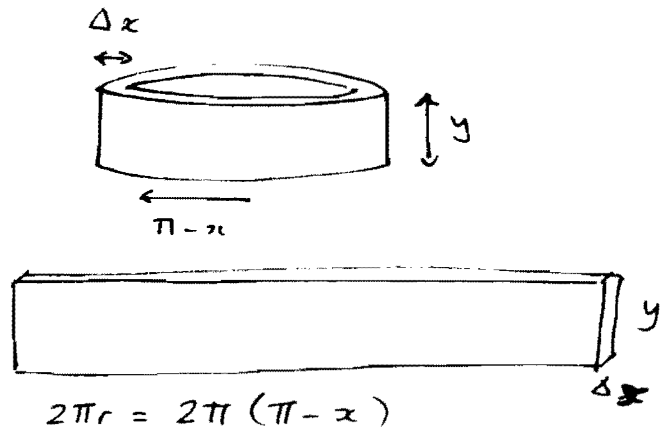
$= \frac{1}{24} \times \frac{\pi}{4}$

$= \frac{\pi}{96}$ $\quad 1$

Question 2.

a)





$$\Delta V = 2\pi\ (\pi - x) \cdot y \cdot \Delta x$$

$$= 2\pi\ (\pi - x) \cdot \cos x \cdot \ \Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{\pi/2} 2\pi\ (\pi - x) \cos x \cdot \Delta x$$

$$= 2\pi \int_0^{\pi/2} \pi \cos x - x \cos x \cdot dx \qquad 1$$

$$= 2\pi \left[\pi \sin x \right]_0^{\pi/2} - \underbrace{2\pi \int_0^{\pi/2} x \cdot \cos x \cdot dx}_{I}$$

$$= 2\pi\ (\pi - 0) - 2\pi \cdot I$$

$$= 2\pi^2 - 2\pi I \qquad 1$$

Now

$$I = \int_0^{\pi/2} x \cos x \cdot dx \qquad u = x \quad u' = 1$$
$$\qquad v = \sin x \quad v' = \cos x$$

$$= \left[x \sin x \right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \cdot dx$$

$$= \left(\frac{\pi}{2} - 0 \right) + \left[\cos x \right]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \cos\frac{\pi}{2} - \cos 0$$

$$= \frac{\pi}{2} + 0 - 1$$

$$= \frac{\pi}{2} - 1 \qquad 1$$

$\therefore \quad V = 2\pi^2 - 2\pi\left(\frac{\pi}{2} - 1\right)$

$= 2\pi^2 - \pi^2 + 2\pi$

$= \underline{\pi^2 + 2\pi \quad \text{unit}^3}$ 1

b) i) $\frac{1}{(5x+3)(x+1)} = \frac{A}{5x+3} + \frac{B}{x+1}$

$1 = A(x+1) + B(5x+3)$

If $x = -1$, $1 = 0 + B(-2)$ $\therefore B = -\frac{1}{2}$

If $x = -\frac{3}{5}$, $1 = A\left(\frac{2}{5}\right) + 0$ $\therefore A = \frac{5}{2}$ 1

$\therefore \int_0^1 \frac{1}{(5x+3)(x+1)} \cdot dx$

$= \int_0^1 \frac{5/2}{5x+3} - \frac{1/2}{x+1} \cdot dx$

$= \left[\frac{1}{2}\ln(5x+3) - \frac{1}{2}\ln(x+1)\right]_0^1$ 1

$= \left(\frac{1}{2}\ln 8 - \frac{1}{2}\ln 2\right) - \left(\frac{1}{2}\ln 3 - \frac{1}{2}\ln 1\right)$

$= \frac{1}{2}\ln 4 - \frac{1}{2}\ln 3$ 1

$= \underline{\frac{1}{2}\ln\frac{4}{3}}$

(ii) $\int_0^{\pi/2} \frac{1}{4\sin x - \cos x + 4} \cdot dx$

$t = \tan\frac{x}{2}$

$2.dt = \cancel{\frac{1}{2}}\sec^2\frac{x}{2} \cdot dx$

$\frac{2.dt}{1+t^2} = 1+t^2 \cdot dx$

If $x = 0$, $t = 0$

If $x = \pi/2$, $t = 1$

1

$= \int_0^1 \frac{1}{\frac{8t}{1+t^2} - \frac{1-t^2}{1+t^2} + 4} \cdot \frac{2}{1+t^2} \cdot dt$

$$= \int_0^1 \frac{2}{8t - 1 + t^2 + 4 + 4t^2} \, dt \qquad 1$$

$$= \int_0^1 \frac{2}{5t^2 + 8t + 3} \, . dt \qquad 1$$

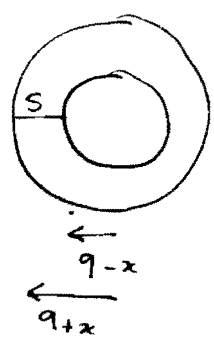
$$= 2 \int_0^1 \frac{1}{(5t+3)(t+1)} \, . dt$$

$= 2 \left(\frac{1}{2} \ln \frac{4}{3} \right)$ from result in (i) 1

$= \underline{\ln 4/3}$

Question 3.

a) i) Endpts of S are (x, y) and $(-x, y)$

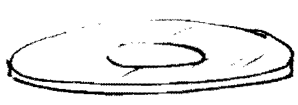


$A = \pi \left((9+x)^2 - (9-x)^2 \right)$ 1

1

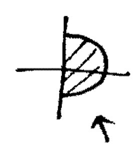
$= \pi \, . \, 36x$ $\left\{ \text{but } x^2 + y^2 = 16 \quad \therefore x = \sqrt{16 - y^2} \right.$ 1

$= 36\pi \, . \sqrt{16 - y^2}$

ii) 

$\Delta V = 36\pi \, . \sqrt{16 - y^2} \, . \, \Delta y$

$V = 36\pi \displaystyle\int_{-4}^{4} \sqrt{16 - y^2} \, . \, dy$ 1



$= 36\pi \times \frac{1}{2} \pi (4)^2$ 1 (area corresponding to integral)

$= 288 \pi^2$ units^3 1

b) i) $$I_n = \int_0^{\pi/4} \tan^n x \, . dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \, . \tan^2 x \, . dx$$ 1

$$= \int_0^{\pi/4} \tan^{n-2} x \, . \, (\sec^2 x - 1) \, . dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \, . \sec^2 x \, . dx - \int_0^{\pi} \tan^{n-2} x \, . dx$$ 1

$$= \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/4} - I_{n-2}$$

$$= \left(\frac{1}{n-1} - 0 \right) - I_{n-2}$$ 1

$$\therefore I_n + I_{n-2} = \frac{1}{n-1}$$

ii) $I_7 + I_5 = \frac{1}{6}$

$I_5 + I_3 = 1/4$

$I_3 + I_1 = 1/2$

$$I_1 = \int_0^{\pi/4} \tan x \, . dx$$

$$= \left[- \ln (\cos x) \right]_0^{\pi/4}$$

$$= - \ln \left(\frac{1}{\sqrt{2}} \right) + \ln 1$$

$$= - \ln \frac{1}{\sqrt{2}}$$

$$= \ln \sqrt{2}$$ 1

$$\therefore I_7 = \frac{1}{6} - I_5$$

$$= \frac{1}{6} - \left(\frac{1}{4} - I_3 \right)$$

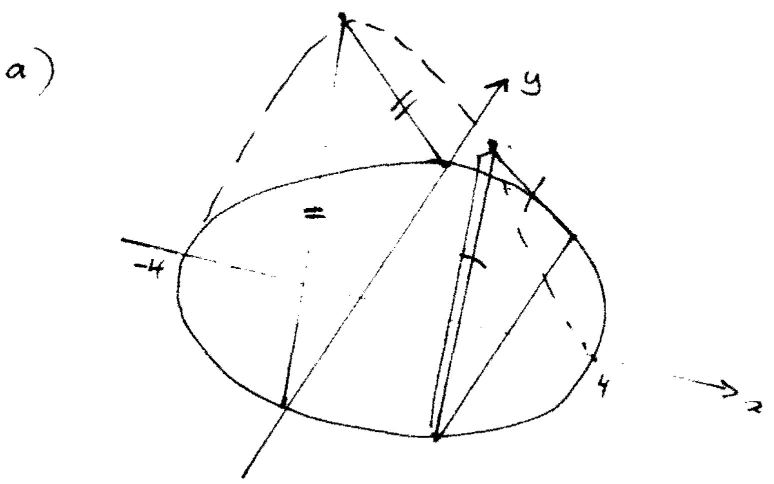
$$= -\frac{1}{12} + I_3$$

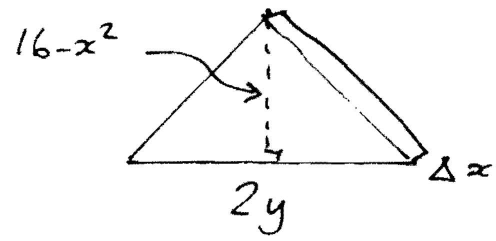
$$= -\frac{1}{12} + \frac{1}{2} - I_1$$

$$= \frac{5}{12} - \ln \sqrt{2}$$ 1

Question 4

a)





(i) $\Delta V = \frac{1}{2} \cdot 2y \cdot (16-x^2) \cdot \Delta x$

$= y(16-x^2) \cdot \Delta x$ but $x^2+y^2=16$, $y = \sqrt{16-x^2}$

$= \sqrt{16-x^2} \cdot (16-x^2) \cdot \Delta x$ $\longleftarrow$ 1

$= (16-x^2)^{3/2} \cdot \Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=-4}^{4} (16-x^2)^{3/2} \cdot dx$

$= \int_{-4}^{4} (16-x^2)^{3/2} \cdot dx$

$= 2\int_{0}^{4} (16-x^2)^{3/2} \cdot dx$ since integrand is an even fn. must mention /

ii) Let $x = 4\sin\theta$ | If $x=0$, $\theta = 0$

$dx = 4\cos\theta \cdot d\theta$ | If $x=4$, $\sin\theta = 1$, $\theta = \frac{\pi}{2}$

$16-x^2 = 16 - 16\sin^2\theta = 16\cos^2\theta$

$V = 2\int_{0}^{4} \sqrt{16\cos^2\theta}^{\,3} \cdot 4\cos\theta \cdot d\theta$ |

$= 2\int_{0}^{4} 64\cos^3\theta \cdot 4\cos\theta \cdot d\theta$

$$= 512 \int_0^{\pi/2} \cos^4 \theta \, . \, d\theta \qquad \cos^4\theta = \left(\tfrac{1}{2}(1 + \cos 2\theta) \right)^2$$

$$= 512 \int_0^{\pi/2} \frac{1}{4} (1 + 2\cos 2\theta + \cos^2 2\theta) \, . \, d\theta \qquad 1$$

$$= 128 \int_0^{\pi/2} 1 + 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta \, . \, d\theta$$

$$= 128 \left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta \right]_0^{\pi/2} \qquad 1$$

$$= 128 \left(\left(\frac{3\pi}{4} + 0 + 0 \right) - 0 \right)$$

$$= \underline{96\pi \text{ units}^3} \qquad 1$$

b) i) $\displaystyle\int_0^{2a} f(x) . dx = \int_0^a f(x) . dx + \int_a^{2a} f(x) . dx$ 1

Now if $u = 2a - x$ then $x = 2a - u$, $dx = -du$

if $x = 2a$, $u = 0$

if $x = a$, $u = a$

$$\therefore \int_a^{2a} f(x) . dx = \int_a^0 f(2a - u) . -du \qquad 1$$

$$= \int_0^a f(2a - u) . du$$

$$= \int_0^a f(2a - x) . dx \qquad 1$$

$$\therefore \text{Original integral} = \int_0^a f(x) + f(2a - x) . dx$$

ii) $\displaystyle\int_0^{\pi} \frac{x}{2+\sin x} \, . dx$ $\qquad \left[f(x) = \frac{x}{2+\sin x} \right]$

$$= \int_0^{\pi/2} f(x) + f(\pi - x) \, . dx$$

$$= \int_0^{\pi/2} \frac{x}{2+\sin x} + \frac{\pi - x}{2+\sin(\pi - x)} \, . dx$$

$$= \int_0^{\pi/2} \frac{x}{2+\sin x} + \frac{\pi - x}{2+\sin x} \, . dx$$

since $\sin(\pi - x) = \sin x$

$$= \int_0^{\pi/2} \frac{\pi}{2+\sin x} \, . dx$$

$$= \pi \int_0^{\pi/2} \frac{1}{2+\sin x} \, . dx$$

$\therefore \boxed{k = \pi}$