BAULKHAM HILLS HIGH SCHOOL

2013

YEAR 12 JUNE ASSESSMENT TASK

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 60minutes
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All relevant mathematical reasoning and/or calculations must be shown

Total marks – 34

There are 4 questions on pages 2-4.

Answer each question in the answer booklet. Each page must show your BOS#. Extra paper is available.

All relevant mathematical reasoning and/or calculations must be shown.

Marks

Question 1 **Start on the relevant page in your answer booklet**

1. a) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx$ **2**

b) i) Find numbers *a*, *b*, and *c* such that **2**

$$\frac{2y+3}{(y-2)(y^2+3)} \equiv \frac{a}{y-2} + \frac{by+c}{y^2+3}$$

ii) Hence find $\displaystyle\int \frac{2y+3}{(y-2)(y^2+3)}\,dy$ **2**

c) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta+\cos\theta}$ **4**

Question 2 **Start on the relevant page in your answer booklet**

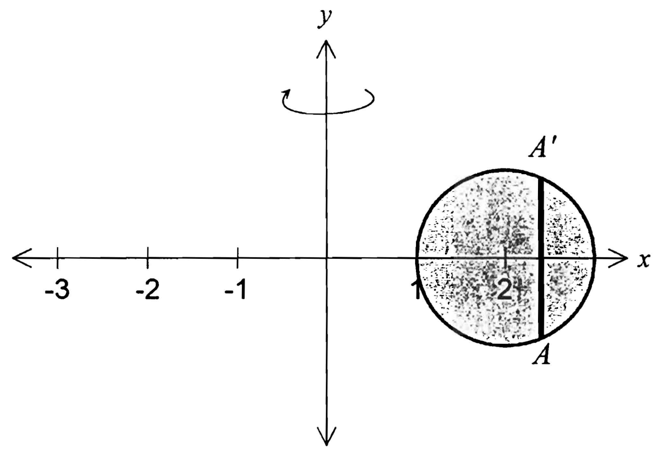
a) i) Prove $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **2**

ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1-\tan x}{1+\tan x}\,dx$ **3**

b) Find $\displaystyle\int \frac{x-2}{\sqrt{8-2x-x^2}}\,dx$ **3**

Question 3 **Start on the relevant page in your answer booklet**

a) The circular region $(x-2)^2 + y^2 \leq 1$ is rotated about the y axis to form a torus.



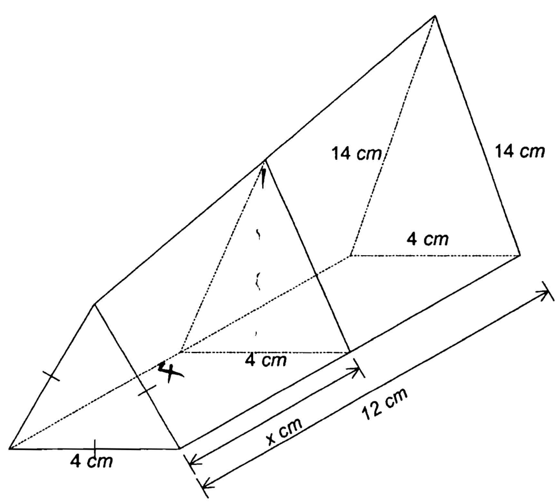
i) Show that the volume ΔV obtained when a typical strip of height AA' and thickness Δx is rotated about the y axis is given by **2**

$$\Delta V = 4\pi x\sqrt{1-(x-2)^2}\ \Delta x$$

ii) Hence find the total volume of the solid generated. **3**

b) The front face of a solid is an equilateral triangle with sides 4 *cm* and theend face is an isosceles triangle with base 4 *cm* and equal sides 14 *cm*.

The solid is 12 *cm* long and cross sections parallel to the front face are isosceles triangles with base 4 *cm.*



i) Show that the height, *h cm,* of a triangular cross section *x cm,* from the front face is given by $h = \frac{\sqrt{3}}{2}(x+4)$ **3**

ii) Hence find the volume of the solid. **2**

Question 4 Start on the relevant page in your answer booklet

Let $I_n = \int_0^1 \frac{x^n}{\sqrt{9 - x^2}}\, dx$ where n is a positive integer.

i) Find the value of I_0 **1**

ii) Show that $nI_n = 9(n-1)I_{n-2} - 2\sqrt{2}$ for $n \geq 2$. **3**

iii) Hence evaluate $\int_0^1 \frac{x^4}{\sqrt{9 - x^2}}\, dx$ **2**

END OF EXAMINATION

① EXTENSION 2 JUNE 2013 SOLUTIONS

1 a) $\int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{x^2+1}}$

let $u = x^2+1$
$du = 2x\,dx$
when $x=\sqrt{3}, u=4$
$x=0, u=1$

2 marks correct answer
1 mk significant progress

$= \frac{1}{2}\int_1^4 \frac{du}{\sqrt{u}}$ ✓

$= \frac{1}{2}\int_1^4 u^{-\frac{1}{2}}\,du$

$= \left[\sqrt{u}\right]_1^4$

$= \sqrt{4} - \sqrt{1}$

$= 1$ ✓

1 b i) let $\frac{2y+3}{(y-2)(y^2+3)} \equiv \frac{a}{y-2} + \frac{by+c}{y^2+3}$

2 - correct answer
1 - one correct or significant progress

$2y+3 \equiv a(y^2+3) + (by+c)(y-2)$

sub $y=2$ $7 = 7a$
$a = 1$

equating coefficients of y^2:
$0 = a+b$
$0 = 1+b$
$b = -1$

of y:
$2 = -2b+c$
$2 = 2+c$
$c = 0$

$\therefore a=1, b=-1, c=0$

ii) $\int \frac{2y+3}{(y-2)(y^2+3)}dy = \int \frac{1}{y-2} - \frac{y}{y^2+3}\,dy$

$= \ln|y-2| - \frac{1}{2}\ln(y^2+3) + C$

2 correct answer
1 - one correct integral

②

1 c. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta+\cos\theta}$

let $t = \tan\frac{\theta}{2}$
$dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,dx$
$2dt = (t^2+1)\,d\theta$
$\therefore d\theta = \frac{2dt}{t^2+1}$ ✓
when $\theta = \frac{\pi}{2}$ $t = \tan\frac{\pi}{4}$
$t = 1$
when $\theta = 0$, $t = \tan 0$
$t = 0$

$= \int_0^1 \frac{2dt}{(1+t^2)\left(1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}\right)}$ ✓

$= \int_0^1 \frac{2dt}{1+t^2+2t+1-t^2}$

$= \int_0^1 \frac{2dt}{2(t+1)}$ ✓

$= \int_0^1 \frac{dt}{t+1}$

$= \left[\ln(t+1)\right]_0^1$

$= \ln 2 - \ln 1$

$= \ln 2$ ✓

2 a i) $\int_0^a f(a-x)\,dx$

let $u = a-x$
$du = -dx$
when $x=a, u=0$
$x=0, u=a$

$= \int_a^0 f(u)(-du)$ ✓

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$ ✓

a ii) $\int_0^{\frac{\pi}{4}} \frac{1-\tan x}{1+\tan x}dx = \int_0^{\frac{\pi}{4}} \frac{1-\tan(\frac{\pi}{4}-x)}{1+\tan(\frac{\pi}{4}-x)}dx$ ✓

$= \int_0^{\frac{\pi}{4}} \frac{1-\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}}{1+\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}}\,dx$

(3)

$$= \int_0^{\pi/4} \frac{1+\tan x - \tan\frac{\pi}{4} + \tan x}{1-\tan x + 1 + \tan x}\,dx$$

$$= \int_0^{\pi/4} \frac{2\tan x}{2}\,dx \quad ✓$$

$$= \Big[-\ln|\cos x|\Big]_0^{\pi/4}$$

$$= -\left(\ln\left(\cos\tfrac{\pi}{4}\right) - \ln\cos 0\right)$$

$$= -\ln\left(\tfrac{1}{\sqrt2}\right) + \ln 1$$

$$= \ln\sqrt2$$

$$= \tfrac12 \ln 2 \quad ✓$$

2b) $\displaystyle\int \frac{x-2}{\sqrt{8-2x-x^2}}\,dx = \int \frac{x-2}{\sqrt{9-(x^2+2x+1)}}\,dx$

$$= \int \frac{x-2}{\sqrt{9-(x+1)^2}}\,dx \quad ✓$$

$$= \frac{-1}{2}\int \frac{-2(x+1)}{\sqrt{9-(x+1)^2}}\,dx - \int \frac{3}{\sqrt{9-(x+1)^2}}\,dx$$

$$= -\frac12 \int \frac{du}{\sqrt u} - 3\sin^{-1}\frac{x+1}{3} + C \quad ✓ \qquad \text{let } u = 9-(x+1)^2,\ du = -2(x+1)\,dx \quad ✓$$

$$= -\sqrt{8-2x-x^2} - 3\sin^{-1}\frac{x+1}{3} + C$$

OR method 2

$$\int \frac{x-2\,dx}{\sqrt{8-2x-x^2}} = \int \frac{x-2\,dx}{\sqrt{9-(x+1)^2}} \quad ✓$$

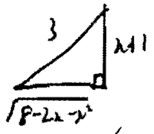
let $1+x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$, $x-2 = 3\sin\theta - 3$

$$= \int \frac{(3\sin\theta-3)\,3\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}}$$

$$= \int \frac{9(\sin\theta-1)\cos\theta\,d\theta}{3\sqrt{1-\sin^2\theta}} \quad ✓$$

$$= 3\int \sin\theta - 1\,d\theta$$

$$= 3(-\cos\theta - \theta) + C$$



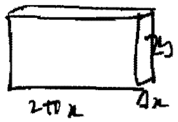
$$= -\sqrt{8-2x-x^2} - 3\sin^{-1}\frac{x+1}{3} + C \quad ✓$$

(4)

3. a) i) $(x-2)^2 + y^2 = 1$

$y^2 = 1-(x-2)^2$

$\therefore AA'$: $2y = 2\sqrt{1-(x-2)^2}$ $(y>0)$ ✓



$\therefore \Delta V = 2\pi x \cdot 2\sqrt{1-(x-2)^2}\,\Delta x$

$= 4\pi x\sqrt{1-(x-2)^2}\,\Delta x$ ✓

ii) $\displaystyle V = \lim_{\Delta x\to 0}\sum_{x=1}^{3} 4\pi x\sqrt{1-(x-2)^2}\,\Delta x$ * must show this line

$\displaystyle = 4\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx$ let $u = x-2$, $x = u+2$, $du = dx$

$\displaystyle = 4\pi\int_{-1}^{1}(u+2)\sqrt{1-u^2}\,du$ ✓ when $x=3$, $u=1$; $x=1$, $u=-1$

$\displaystyle = 4\pi\int_{-1}^{1}\left(u\sqrt{1-u^2} + 2\sqrt{1-u^2}\right)du$

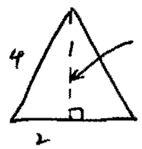
(odd function) (semicircle)

$\displaystyle = 4\pi\left[0 + 2\int_{-1}^{1}\sqrt{1-u^2}\,du\right]$ ✓

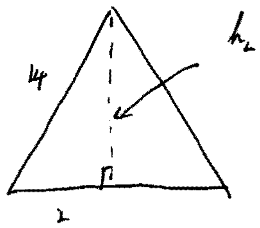
$= 4\pi \times 2 \times \frac{\pi \times 1^2}{2}$

Volume $= 4\pi^2$ units^3 ✓

3 b)



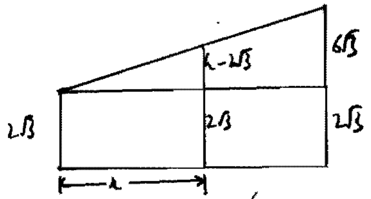
$h_r = \sqrt{16-4}$
$= 2\sqrt3$



$h_L = \sqrt{14^2 - 2^2}$
$= 8\sqrt3$

✓

Method 1 Similarity



$\frac{h-2\sqrt3}{6\sqrt3} = \frac{x}{12}$ as triangles are similar

$h - 2\sqrt3 = \frac{x\sqrt3}{2}$

$h = \frac{x\sqrt3}{2} + 2\sqrt3$

$= \frac{\sqrt3}{2}(x+4)$

Method 2 Linear functions

let $h = ax + b$

when $x=0$ $h = 2\sqrt3$

$2\sqrt3 = 0 + b$

$\therefore b = 2\sqrt3$

when $x = 12$, $h = 8\sqrt3$

$8\sqrt3 = 12a + 2\sqrt3$

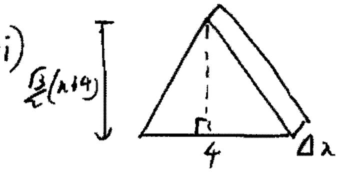
$6\sqrt3 = 12a$

$a = \frac{\sqrt3}{2}$

$\therefore h = \frac{x\sqrt3}{2} + 2\sqrt3$

$= \frac{\sqrt3}{2}(x+4)$

ii)



Volume of slice $\Delta V = \frac{1}{2} \times 4 \times \frac{\sqrt3}{2}(x+4)\,\Delta x$

$\Delta V = \sqrt3(x+4)\,\Delta x$

Volume $= \lim_{\Delta x \to 0} \sum_{x=0}^{12} \sqrt3(x+4)\Delta x$

$= \sqrt3 \int_0^{12} x + 4 \, dx$

$= \sqrt3 \left[\frac{x^2}{2} + 4x\right]_0^{12}$

$= \sqrt3(72 + 48 - 0)$

$\therefore$ Volume $= 120\sqrt3$ cm^3

4 i) $I_0 = \int_0^1 \frac{x^0}{\sqrt{9-x^2}}\,dx$

$= \int_0^1 \frac{1}{\sqrt{9-x^2}}\,dx$

$= \left[\sin^{-1}\frac{x}{3}\right]_0^1$

$I_0 = \sin^{-1}\frac{1}{3}$

4 ii) $I_n = \int_0^1 \frac{x^n}{\sqrt{9-x^2}}\,dx$

$= \int_0^1 \frac{x \cdot x^{n-1}}{\sqrt{9-x^2}}\,dx$

$u = x^{n-1}$ $\quad v' = \frac{x}{\sqrt{9-x^2}}$

$u' = (n-1)x^{n-2}$ $\quad v = -\sqrt{9-x^2}$

$I_n = \left[-x^{n-1}\sqrt{9-x^2}\right]_0^1 + (n-1)\int_0^1 x^{n-2}\sqrt{9-x^2}\,dx$

$= (-\sqrt8 - 0) + (n-1)\int_0^1 \frac{x^{n-2}(9-x^2)}{\sqrt{9-x^2}}\,dx$

$I_n = -2\sqrt2 + (n-1)\int_0^1 \frac{9x^{n-2} - x^n}{\sqrt{9-x^2}}\,dx$

$I_n = -2\sqrt2 + (n-1)\int_0^1 \frac{9x^{n-2}}{\sqrt{9-x^2}}\,dx - (n-1)\int_0^1 \frac{x^n}{\sqrt{9-x^2}}\,dx$

$I_n = -2\sqrt2 + 9(n-1)\int_0^1 \frac{x^{n-2}}{\sqrt{9-x^2}}\,dx - (n-1)I_n$

$I_n(1 + n - 1) = 9(n-1)I_{n-2} - 2\sqrt2$

$nI_n = 9(n-1)I_{n-2} - 2\sqrt2$

4 iii) $I_4 = \int_0^1 \frac{x^4}{\sqrt{9-x^2}}\,dx$

$4I_4 = 9 \times 3\, I_2 - 2\sqrt2$

$4I_4 = \frac{27}{2} 2I_2 - 2\sqrt2$

$= \frac{27}{2}(9I_0 - 2\sqrt2) - 2\sqrt2$

$4I_4 = \frac{27}{2}\left(9\sin^{-1}\left(\frac13\right) - 2\sqrt2\right) - 2\sqrt2$

$I_4 = \frac{27}{8}\left(9\sin^{-1}\left(\frac13\right) - 2\sqrt2\right) - \frac{\sqrt2}{2}$

$I_4 = \frac{243}{8}\sin^{-1}\left(\frac13\right) - \frac{29\sqrt2}{4}$