BAULKHAM HILLS HIGH SCHOOL

2014
YEAR 12
TERM 2 ASSESSMENTS

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 70 minutes
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All relevant mathematical reasoning and/or calculations must be shown.

Total marks – 35

- Attempt all Questions in the booklet provided.

35 marks
Attempt All Questions

Answer each question on the appropriate answer sheet. Each answer sheet must show your BOS#. Extra paper is available.

Your responses should include relevant mathematical reasoning and/or calculations.

Marks

Question 1 ***(8 marks)*** Use a *separate* answer sheet

(a) Each of the following statements is either true or false. Write TRUE or FALSE for each statement and give brief reasons for your answers. (You are NOT required to evaluate the integrals)

(i) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^5 x\, dx = 0$ *1*

(ii) $\displaystyle\int_0^{\pi} \cos^7 x\, dx > 0$ *1*

(iii) $\displaystyle\int_0^1 e^{-x^2}\, dx = 0$ *1*

(iv) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^5}} > \int_0^1 \frac{dx}{\sqrt{1+x^6}}$ *1*

(b) Explain the error in the following solution: *1*

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x dx}{\cos x} \\
&= \int \sec x \sin x dx \\
&= -\sec x \cos x + \int \sec x \tan x \cos x dx \\
&= -\sec x \cos x + \int \tan x dx \\
\int \tan x dx - \int \tan x dx &= -\sec x \cos x \\
0 &= -1
\end{aligned}$$

(c) By using the relationship $\int_0^a f(a-x)\, dx = \int_0^a f(x)\, dx$, or otherwise, evaluate *3*

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$$

Marks

<u>Question 2</u> ***(12 marks)*** Use a *separate* answer sheet

(a) Find the following integrals

(i) $\displaystyle\int \cos^2 x \cot x \, dx$ *3*

(ii) $\displaystyle\int \frac{dx}{\sqrt{3 + 2x + x^2}}$ *2*

(iii) $\displaystyle\int x \tan^{-1} x \, dx$ *3*

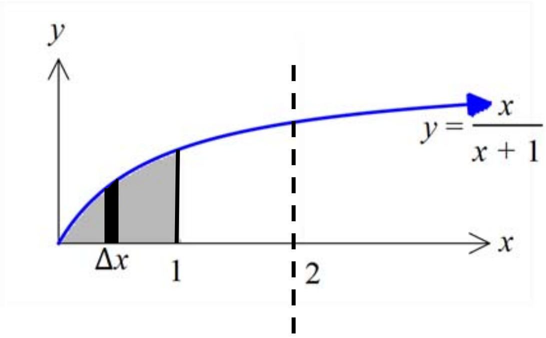
(b) (i) Find numbers *A*, *B*, *C* and *D* such that *2*

$$\frac{2x^3 - 5x^2 + 6x - 3}{x^4 + 3x^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 3}$$

(ii) Hence find $\displaystyle\int \frac{2x^3 - 5x^2 + 6x - 3}{x^4 + 3x^2} dx$ *2*

<u>Question 3</u> ***(4 marks)*** Use a *separate* answer sheet

The shaded region shown below, bounded by the portion of the curve $y = \dfrac{x}{x+1}$, the x-axis and the line $x = 1$ is rotated about the line $x = 2$.



(i) Using the method of cylindrical shells, show that volume ΔV of a typical shell at a distance x from the origin and thickness Δx is given by *1*

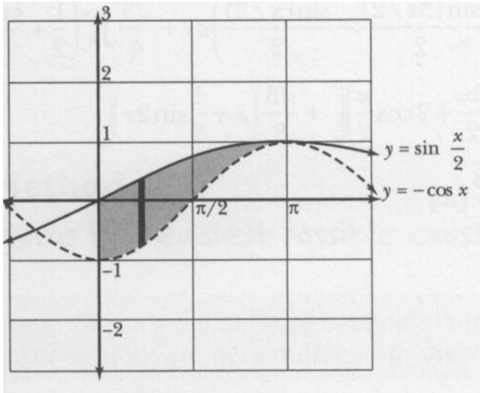
$$\Delta V = \frac{2\pi x(2 - x)}{1 + x} \Delta x$$

(ii) Hence find the volume of the solid. *3*

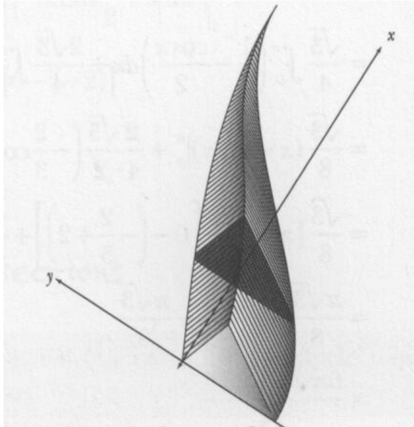
Marks

<u>Question 4</u> ***(5 marks)*** Use a *separate* answer sheet

The base of a certain solid, below right, is bounded by the graphs $y = -\cos x$, $y = \sin\dfrac{x}{2}$ and the y-axis as shown in the diagram of Region G below left.



Region G



Solid with base G

(i) Show that the curves $y = -\cos x$ and $y = \sin\dfrac{x}{2}$ intersect at the point $(\pi, 1)$. *1*

(ii) Show that an equilateral triangle, with side length s, has an area $\dfrac{s^2\sqrt{3}}{4}\ units^2$ *1*

(iii) Calculate the volume of the solid with base Region G whose cross-sections are equilateral triangles perpendicular to the x-axis. *3*

<u>Question 5</u> ***(6 marks)*** Use a *separate* answer sheet

Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,dx$

(i) Show that $I_n = \left(\dfrac{n-1}{n}\right) I_{n-2}$, for $n \geq 2$ *3*

(ii) Hence show that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \dfrac{\pi(2n)!}{2^{2n+1}(n!)^2}$ *3*

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \ n \neq -1; x \neq 0, \text{ if } \ n < 0$$

$$\int \frac{1}{x} dx = \ln x, \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \ a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \ \ x > a > 0$$

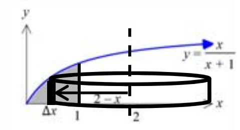
$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

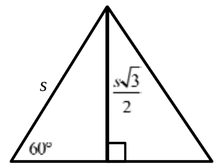
NOTE: $\ln x = \log x, \quad x > 0$

BAULKHAM HILLS HIGH SCHOOL
YEAR 12 EXTENSION 2 TERM 2 ASSESSMENTS 2014 SOLUTIONS

Solution	Marks	Comments
QUESTION 1		
1(a) (i) TRUE $\tan x$ is an odd function and an $(\text{odd function})^5$ is an odd function, thus $\tan^5 x$ has symmetry about the origin, and so the given integral = 0	**1**	**1 mark** • Correct answer with a logical explanation
1(a) (ii) FALSE $\cos x$ has rotational symmetry about $\left(\frac{\pi}{2},0\right)$ and thus so does $\cos^7 x$. This means the area from 0 to $\frac{\pi}{2}$ is above the x-axis and is equal to the area from $\frac{\pi}{2}$ to π, which is below the x-axis, and so the given integral = 0	**1**	**1 mark** • Correct answer with a logical explanation
1(a) (iii) FALSE The curve $y = e^{-x^2}$ is above the x-axis for all values of x, and so the given integral must be > 0	**1**	**1 mark** • Correct answer with a logical explanation
1(a) (iv) FALSE For the domain $0 < x < 1$; $0 < 1 + x^6 < 1 + x^5$ $0 < \sqrt{1+x^6} < \sqrt{1+x^5}$ $\frac{1}{\sqrt{1+x^6}} > \frac{1}{\sqrt{1+x^5}} > 0$ $\therefore \int_0^1 \frac{dx}{\sqrt{1+x^6}} > \int_0^1 \frac{dx}{\sqrt{1+x^5}} > 0$	**1**	**1 mark** • Correct answer with a logical explanation
1(b) $\int \tan x\,dx = -\sec x \cos x + c + \int \tan x dx$ $\int \tan x dx - \int \tan x dx = -\sec x \cos x + c$ $0 = -1 + c$ i.e. the error is neglecting the constant of integration	**1**	**1 mark** • Correct explanation
1(c) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$ $= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+(-\cos x)^2}dx$ $= \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ $\therefore 2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$ $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$ $u = \cos x$, $du = -\sin x\,dx$ $= -\frac{\pi}{2}\int_1^{-1} \frac{du}{1+u^2}$ $x = 0,\ u = 1$; $x = \pi,\ u = -1$ $= \frac{\pi}{2}\left[\tan^{-1} u\right]_{-1}^{1}$ $= \frac{\pi}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$ $= \frac{\pi^2}{4}$	**3**	**3 marks** • Correct solution **2 marks** • Uses trig relationships to rearrange expression into a manageable integrand. **1 mark** • Successfully uses $\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$ .or equivalent merit

Solution	Marks	Comments
QUESTION 2		
2(a) (i) $\int \cos^2 x \cot x\, dx = \int \frac{\cos^3 x\, dx}{\sin x}$ $u = \sin x$, $du = \cos x\, dx$ $= \int \frac{(1 - \sin^2 x)\cos x\, dx}{\sin x}$ $= \int \left(\frac{1}{u} - u\right) du$ $= \ln\lvert u\rvert - \frac{1}{2}u^2 + c$ $= \ln\lvert \sin x\rvert - \frac{1}{2}\sin^2 x + c$	**3**	**3 marks** • Correct solution **2 marks** • Obtains the correct primitive in terms of the substituted variable **1 mark** • Makes a valid substitution.
2(a) (ii) $\int \frac{dx}{\sqrt{3+2x+x^2}} = \int \frac{dx}{\sqrt{2+(x+1)^2}}$ $= \ln\left\lvert x + 1 + \sqrt{3+2x+x^2}\right\rvert + c$	**2**	**2 marks** • Correct answer **1 mark** • Completes the square in the denominator • Correctly uses standard integral for their denominator, after completing the square
2(a) (iii) $\int x\tan^{-1} x\, dx$ $u = \tan^{-1} x$, $v = \frac{1}{2}x^2$; $du = \frac{dx}{1+x^2}$, $dv = x\,dx$ $= \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\int \frac{x^2 dx}{1+x^2}$ $= \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx$ $= \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + c$	**3**	**3 marks** • Correct solution **2 marks** • Significant progress towards the correct solution. **1 mark** • Uses integration by parts to rewrite the integrand.
2(b) (i) $\frac{2x^3 - 5x^2 + 6x - 3}{x^4 + 3x^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 3}$ $Ax(x^2+3) + B(x^2+3) + (Cx+D)x^2 \equiv 2x^3 - 5x^2 + 6x - 3$ $x = 0$: $3B = -3$, $B = -1$ coefficient of x: $3A = 6$, $A = 2$ $x = i\sqrt{3}$: $-3iC\sqrt{3} - 3D = -6i\sqrt{3} + 15 + 6i\sqrt{3} - 3$ $= 12$ $\therefore$ $C = 0$ and $D = -4$ $A = 2,\ B = -1,\ C = 0,\ D = -4$	**2**	**2 marks** • Correct answer **1 mark** • Makes progress towards finding values using correct methods
2(b) (ii) $\int \frac{2x^3 - 5x^2 + 6x - 3}{x^4 + 3x^2}dx = \int \left(\frac{2}{x} - \frac{1}{x^2} - \frac{4}{x^2+3}\right)dx$ $= 2\ln\lvert x\rvert + \frac{1}{x} - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$	**2**	**2 marks** • Correct solution *using their values from (i)* **1 mark** • Finds two correct primitives obtained from their integrand

Solution	Marks	Comments
QUESTION 3		
3 (i)  $y=\frac{x}{x+1}$  $2\pi(2-x)$ $A(x)=2\pi(2-x)\times\frac{x}{x+1}$ $\Delta V=\frac{2\pi x(2-x)}{x+1}$	1	**1 mark** • Derives the correct expression **without** just substituting into a formula such as $2\pi xy$
3 (ii) $V=\lim_{\Delta x\to 0}\sum_{x=0}^{1}\frac{2\pi x(2-x)}{x+1}\Delta x$ $=2\pi\int_0^1\frac{2x-x^2}{x+1}dx$ $=2\pi\int_0^1\left(3-x-\frac{3}{x+1}\right)dx$ $=2\pi\left[3x-\frac{x^2}{2}-3\ln\lvert x+1\rvert\right]_0^1$ $=2\pi\left(3-\frac{1}{2}-3\ln 2-0\right)$ $=\pi(5-6\ln 2)\,units^3$	3	**3 marks** • Correct solution **2 marks** • Finds the primitive obtained from a correct integrand after expressing the volume as a limiting sum **1 mark** • Expresses volume as a limiting sum • Finds the primitive obtained from a correct integrand

Solution	Marks	Comments
QUESTION 4		
4 (i) when $x=\pi$, $-\cos x=-\cos\pi$ $=-(-1)$ $=1$ $\sin\frac{x}{2}=\sin\frac{\pi}{2}$ $=1$ ∴ the two curves intersect at the point $(\pi,1)$	1	**1 mark** • Shows correctly
4 (ii) $A=\frac{1}{2}\text{bh}$ $=\frac{1}{2}\times s\times\frac{s\sqrt{3}}{2}$ $=\frac{s^2\sqrt{3}}{4}$ 	1	**1 mark** • Correct solution
4 (iii) $A(x)=\frac{\sqrt{3}}{4}\left(\sin\frac{x}{2}+\cos x\right)^2$ $\Delta V=\frac{\sqrt{3}}{4}\left(\sin\frac{x}{2}+\cos x\right)^2\Delta x$ $V=\lim_{\Delta x\to 0}\sum_{x=0}^{\pi}\frac{\sqrt{3}}{4}(\sin\frac{x}{2}+\cos x)^2\,\Delta x$ $V=\frac{\sqrt{3}}{4}\int_0^{\pi}(\sin\frac{x}{2}+\cos x)^2\,dx$ $=\frac{\sqrt{3}}{4}\int_0^{\pi}\left(\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos x+\cos^2 x\right)dx$ $=\frac{\sqrt{3}}{4}\int_0^{\pi}\left(\frac{1}{2}(1-\cos x)+2\sin\frac{x}{2}\left(2\cos^2\frac{x}{2}-1\right)+\frac{1}{2}(1+\cos 2x)\right)dx$ $=\frac{\sqrt{3}}{4}\int_0^{\pi}\left(1-\frac{1}{2}\cos x+4\cos^2\frac{x}{2}\sin\frac{x}{2}-2\sin\frac{x}{2}+\frac{1}{2}\cos 2x\right)dx$ $=\frac{\sqrt{3}}{4}\left[x-\frac{1}{2}\sin x-\frac{8}{3}\cos^3\frac{x}{2}+4\cos\frac{x}{2}+\frac{1}{4}\sin 2x\right]_0^{\pi}$ $=\frac{\sqrt{3}}{4}\left(\pi-0-0+0+0-0+0+\frac{8}{3}-4-0\right)$ $=\frac{\sqrt{3}}{12}(3\pi-4)\,units^3$	3	**3 marks** • Correct solution **2 marks** • Substantial progress towards a solution using logical techniques. **1 mark** • Expresses area of cross-sectional face in terms of x • Uses trig identities in an attempt to obtain a manageable integrand

Solution	Marks	Comments
QUESTION 5		
5 (i) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$ $u = \sin^{n-1} x$ $du = (n-1)\sin^{n-2} x\cos x dx$ $v = -\cos x$ $dv = \sin x\, dx$ $= \left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\, dx$ $= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x(1-\sin^2 x)dx$ $= (n-1)\int_0^{\frac{\pi}{2}} \left(\sin^{n-2} x - \sin^n x\right)dx$ $= (n-1)I_{n-2} - (n-1)I_n$ $\therefore nI_n = (n-1)I_{n-2}$ $I_n = \left(\frac{n-1}{n}\right)I_{n-2}$	3	**3 marks** • Correct solution **2 marks** • Substantial progress towards a solution using logical techniques **1 mark** • Attempts to create the reduction formula by using integration by parts, or similar merit.
5 (ii) $\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = I_{2n}$ $= \frac{2n-1}{2n} I_{2n-2}$ $= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} I_{2n-4}$ $= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} I_{2n-6}$ $= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \ldots \times \frac{1}{2} I_0$ $= \frac{(2n-1)(2n-3)(2n-5)\ldots(5)(3)(1)}{2(n)2(n-1)2(n-2)\ldots2(3)2(2)2(1)} I_0$ $= \frac{(2n-1)(2n-3)(2n-5)\ldots(5)(3)(1)}{2^n n!} I_0 \times \frac{2n(2n-2)(2n-4)\ldots(6)(4)(2)}{2n(2n-2)(2n-4)\ldots(6)(4)(2)}$ $= \frac{2n(2n-1)(2n-2)(2n-3)\ldots(3)(2)(1)}{2^n n! \times 2^n n!} I_0$ $= \frac{(2n)!}{2^{2n}(n!)^2}\int_0^{\frac{\pi}{2}} dx$ $= \frac{(2n)!}{2^{2n}(n!)^2}\left[x\right]_0^{\frac{\pi}{2}}$ $= \frac{(2n)!}{2^{2n}(n!)^2} \times \frac{\pi}{2}$ $= \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$	3	**3 marks** • Correct solution **2 marks** • Substantial progress towards a solution using logical techniques **1 mark** • Uses the reduction formula to reduce to a manageable integrand. • Evaluates I_0