BAULKHAM HILLS HIGH SCHOOL

**2016**

**YEAR 12 June - Task 3**

# Mathematics Extension 2

**General Instructions**

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Attempt all questions
- Start a new page for each question

**Total marks – 34**
**Exam consists of 4 pages.**

**Standard integrals provided on page 5**

**Question 1 (11 marks) Start on the appropriate page of your answer booklet.** **Marks**

a) Find the indefinite integrals:

i) $\displaystyle\int \frac{dx}{\sqrt{x^2+6x+13}}$ **2**

ii) $\displaystyle\int \frac{x+3}{x^2+9}\,dx$ **2**

iii) $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx$ **3**

b) i) Find the numbers $a, b$ and $c$ such that **2**

$$\frac{x-6}{(x+4)(x-1)^2}=\frac{a}{x+4}+\frac{b}{x-1}+\frac{c}{(x-1)^2}$$

ii) Hence find **2**

$$\int \frac{x-6}{(x+4)(x-1)^2}\,dx$$

**Question 2** **(13 marks) Start on the appropriate page of your answer booklet.** **Marks**

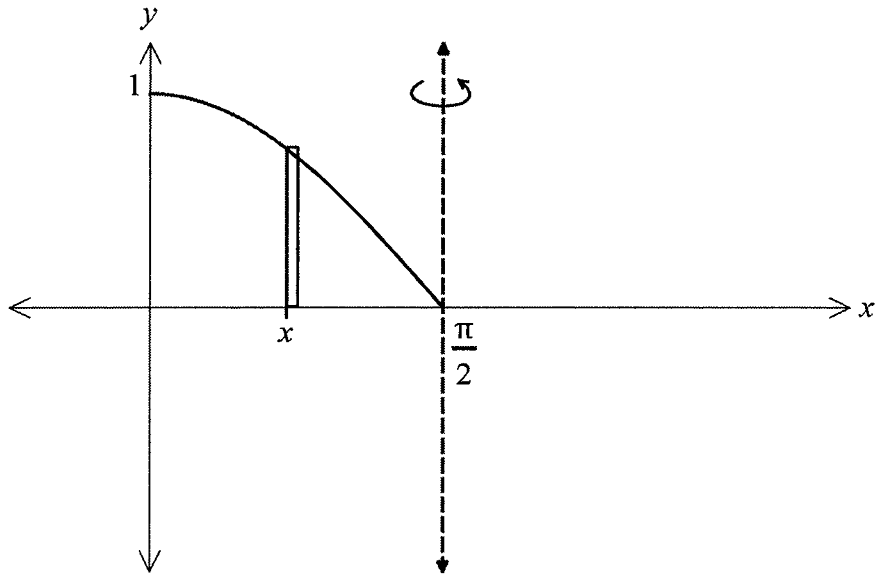
a) Evaluate: **4**

$$\int_1^{\sqrt{3}} \frac{dx}{x\sqrt{1+x^2}}$$

b) i) Prove $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$ **2**

ii) Hence evaluate $\int_0^\pi \frac{x \sin x}{1+\cos^2 x}dx$. **3**

c) The region in the first quadrant bounded by $y = \cos x$, $x$-axis and the $y$-axis is rotated about the line $x = \frac{\pi}{2}$.



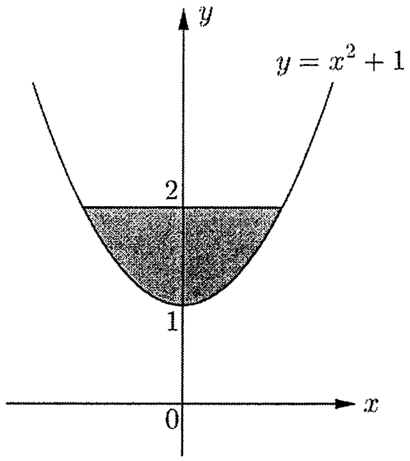

i) Using the method of cylindrical shells, show that the volume, $\Delta V$, of a typical shell at a distance $x$ from the origin and thickness $\Delta x$ is given by **1**

$$\Delta V = 2\pi\left(\frac{\pi}{2} - x\right)(\cos x)\Delta x$$

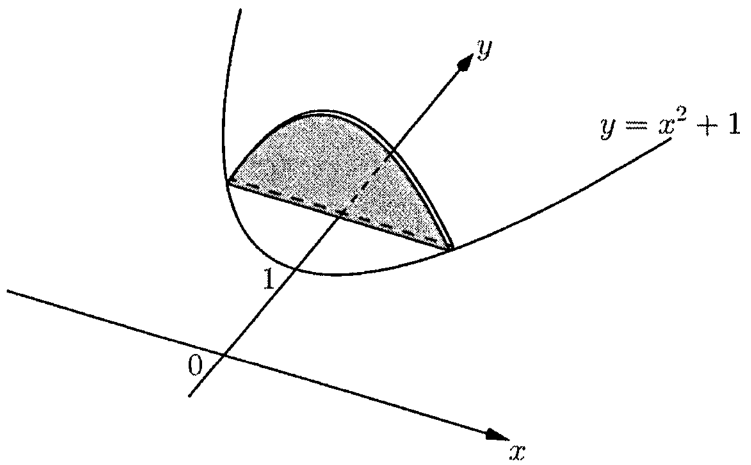
ii) Hence find the volume of this solid. **3**

| **Question 3** (10 marks) **Start on the appropriate page of your answer booklet.** | **Marks** |
|---|---|

a) The base of a solid, $S$, is the region enclosed by the parabola $y = x^2 + 1$ and the line $y = 2$.




Each cross section of $S$, perpendicular to the $y$-axis is another parabola. The height of each cross section that is $y$ units from the origin is $\frac{1}{2}y$ units.




(i) Using Simpson's Rule, show that the area of each cross section $y$ units from the origin is given by $\frac{2}{3}y\sqrt{y-1}$ square units. **2**

(ii) Hence find the volume of the solid $S$. **3**

b) If $I_n = \int_0^1 x^n(x-1)^6 dx$ for $n \geq 0$

(i) Show that $I_n = \dfrac{n}{n+7} I_{n-1}$ for $n \geq 1$ **3**

(ii) Hence find $\int_0^1 x^3(x-1)^6\, dx$ **2**

**-- End of Exam --**

# X2 - 2016 June - Answers

## $\boxed{\text{Question 1}}$

a) i) $\displaystyle\int \frac{dx}{\sqrt{x^2+6x+13}} = \int \frac{dx}{\sqrt{x^2+6x+9+4}} = \int \frac{dx}{\sqrt{(x+3)^2+2^2}}$ (1)

$= \ln\left(x+3+\sqrt{(x+3)^2+4}\right) = \ln\left(x+3+\sqrt{x^2+6x+13}\right)+C$ (1)

ii) $\displaystyle\int \frac{x+3}{x^2+9}\,dx = \int \frac{x}{x^2+9} + \frac{3}{x^2+9}\,dx$

$= \frac{1}{2}\ln(x^2+9) + \tan^{-1}\frac{x}{3} + C$

(1) (1)

iii) $\displaystyle\int \frac{\sin^3 x\,dx}{\cos^2 x} = \int \frac{(1-\cos^2 x)\sin x}{\cos^2 x}\,dx = \int \frac{\sin x}{\cos^2 x} - \sin x\,dx$

$= \int \tan x . \sec x\,dx - \int \sin x\,dx$

$= \sec x + \cos x + C$

~~OR~~ $= \dfrac{\sin^2 x}{\cos x} + 2\sec x + C$ (using IBP)

~~OR~~ $\cancel{\sin x \tan x + 2\cos x + C}$

3- correct solns.
2- correct approach and one of the integrant correct
1- significant progress towards solns.

b) i) $\dfrac{x-6}{(x+4)(x-1)^2} = \dfrac{a}{x+4} + \dfrac{b}{x-1} + \dfrac{c}{(x-1)^2}$

$x-6 = a(x-1)^2 + b(x+4)(x-1) + c(x+4)$

$x=1 \therefore -5 = c(5) \therefore \boxed{c=-1}$

$x=-4 \therefore -10 = a(25) \therefore \boxed{a=-2/5}$

$0x^2 = ax^2 + bx^2$

$0 = -\frac{2}{5} + b \therefore \boxed{b = 2/5}$

2- all answers correct
1- two of the three answers correct

Q1. b) ii) $\displaystyle\int \frac{x-6}{(x+4)(x-1)^2}\,dx = \int \frac{-\frac{2}{5}}{x+4} + \frac{\frac{2}{5}}{x-1} - \frac{1}{(x-1)^2}\,dx$

$= -\frac{2}{5}\ln(x+4) + \frac{2}{5}\ln(x-1) - \frac{(x-1)^{-1}}{-1} + C$

$= \frac{2}{5}\ln\frac{x-1}{x+4} + \frac{1}{x-1} + C$

2- correct soln
1- applying part. correctly

## Question 2 (13 marks)

a) $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{x\left(\sqrt{1+x^2}\right)}$

let $x = \tan\theta$
$1+x^2 = \sec^2\theta$
$\frac{dx}{d\theta} = \sec^2\theta$

$x=1 \therefore \theta = \frac{\pi}{4}$
$x=\sqrt{3}$ $\theta = \frac{\pi}{3}$

$= \displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2\theta\,d\theta}{\tan\theta . \sqrt{\sec^2\theta}}$ (1)

$= \displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec\theta\,d\theta}{\tan\theta} = \int_{\pi/4}^{\pi/3} \operatorname{cosec}\theta\,d\theta$ (1)

$= \Big[-\ln(\operatorname{cosec}\theta + \cot\theta)\Big]_{\pi/4}^{\pi/3}$ (1)

$= -\ln\left[\underbrace{\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}}\right] + \ln\left[\sqrt{2}+1\right]$

$\frac{3}{\sqrt{3}} = \sqrt{3}$

$= \ln\dfrac{\sqrt{2}+1}{\sqrt{3}}$ (1)

4- correct solns
3- correct subst. and limits, simplifying to $\int$ cosec and progress towards solution
2- correct subst. & limits and simpl. to $\int$ cosec θ or progress towards soln
1- correct subst. and limits

Q.2 b) i) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

RHS $= \int_0^a f(a-x)\,dx$ let $u = a-x$

$du = -dx$

$x=0 \therefore u=a$

$x=a \therefore u=0$

$= \int_{u=a}^{0} f(u)\cdot(-du)$

$= -\int_a^0 f(u)\,du$

2 marks - correct proof
1 mark - correct subst and limits

$= \int_0^a f(u)\,du = \int_0^a f(x)\,dx = LHS \therefore$ proven

ii) $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{(\pi-x)\cdot\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$

$\therefore \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{(\pi-x)\cdot\sin x}{1+(-\cos x)^2}\,dx$

$\therefore I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}\,dx - \underbrace{\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx}_{I}$ 1

$\therefore 2I = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$ 1

let $u = \cos x$
$du = -\sin x\,dx$
$x=0 \therefore u=1$
$x=\pi \therefore u=-1$

$\therefore 2I = \pi\int_1^{-1} \frac{-du}{1+u^2} = \pi\int_{-1}^{1} \frac{du}{1+u^2}$

$2I = \pi\left[\tan^{-1} u\right]_{-1}^{1} = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}$

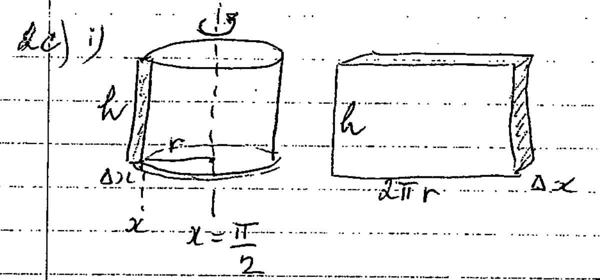
$I = \frac{\pi^2}{4}$ 1

2 b ii) cont:

$\therefore \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$

3 - correct solns
2 - correctly applying the rule and finding a correct integral for $2I$
1 - correctly applying the rule and simplifying the integral

2c) i)




where $h = y = \cos x$
$r = \frac{\pi}{2} - x$

1

$\therefore \Delta V = 2\pi r h\cdot\Delta x$

$\Delta V = 2\pi\left(\frac{\pi}{2} - x\right)\times(\cos x)\,\Delta x$

ii) $V \doteq \lim_{\Delta x \to 0} \sum_{x=0}^{\pi/2} 2\pi\left(\frac{\pi}{2} - x\right)(\cos x)\,\Delta x$

$\therefore V = 2\pi\int_0^{\pi/2} \left(\frac{\pi}{2} - x\right)\cdot\cos x\,dx$ 1

3 - correct solns.
2 - correctly integrates by parts
1 - significant progress towards solns.

$\therefore V = 2\pi\left[\int_0^{\pi/2} \frac{\pi}{2}\cos x\,dx - \int_0^{\pi/2} x\cdot\cos x\,dx\right]$

$\therefore V = 2\pi\left[\frac{\pi}{2}\sin x\right]_0^{\pi/2} - 2\pi\int_0^{\pi/2} \underbrace{x}_{u}\cdot\underbrace{\cos x}_{v'}\,dx$ 1

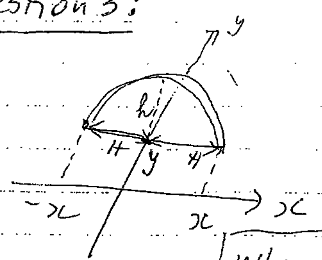
let
$u = x \quad u' = 1$
$v' = \cos x \quad v = \sin x$

$\therefore V = \pi^2[1-0] - 2\pi\left[x\cdot\sin x\right]_0^{\frac{\pi}{2}} + 2\pi\int_0^{\pi/2} \sin x\,dx$

$= \pi^2 - 2\pi\left[\frac{\pi}{2} - 0\right] + 2\pi\left[-\cos x\right]_0^{\pi/2} = -2\pi[0-1] = 2\pi$ 1

Question 3:

a)




Each cross-section = parabola

i) $\therefore$ $\text{Area}_{\text{parabola}} = \frac{H}{3}\left(0 + 4\times\frac{1}{2}y + 0\right)$ (1)

$\boxed{\text{where } h = \frac{1}{2}y}$ $\boxed{\text{and } H = x}$ (1) length of each strip

$\therefore \text{Area}_{\text{parabola}} = \frac{x}{3}\times 2y$

but $y = x^2+1$ $\therefore$ $x = \pm\sqrt{y-1}$ but $H>0$

$\therefore x = +\sqrt{y-1}$ $\because x>0$ length

$\therefore \text{Area}_{\text{parabola}} = \frac{\sqrt{y-1}}{3}\times 2y = \frac{2}{3}y\sqrt{y-1}$ $\therefore$ shown

ii) $\Delta V = \text{Area}_{\text{cross-section}} \times \Delta y = \frac{2}{3}y\sqrt{y-1}\cdot\Delta y$

$\therefore V = \lim_{\Delta y\to 0}\sum_{y=1}^{y=2}\frac{2}{3}y\sqrt{y-1}\cdot\Delta y$ (1) $= \frac{2}{3}\int_1^2 y\sqrt{y-1}\,dy$

$\therefore V = \frac{2}{3}\int_1^2 y\sqrt{y-1}\,dy$ let $u = y-1$ $\therefore du = dy$

$y = u+1$ $y=1 \therefore u=0$

$y=2 \therefore u=1$

(*) $V = \frac{2}{3}\int_0^1 (u+1)\sqrt{u}\,du$ (1) OR $\frac{2}{3}\int_1^2 (y-1)\sqrt{y-1} + \sqrt{y-1}\,dy$

$V = \frac{2}{3}\int_0^1 u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du = \frac{2}{3}\left[\frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3}\right]_0^1$

$= \frac{2}{3}\left[\frac{2}{5} + \frac{2}{3} - 0\right] = \frac{32}{45}$ (1) $(\text{units}^3)$

(*) OR by parts $\frac{2}{3}\left[\frac{2}{3}u(u-1)^{\frac{3}{2}}\right]^2 - \frac{2}{3}\int 1\cdot\frac{2}{3}(u-1)^{\frac{3}{2}}du$

| Marks | |
|---|---|
| 3a) 3 | correct solns. |
| 2 | deriving the volume formula and progress towards solns |
| 1 | correctly derives the volume formula |

3b) i) show $I_n = \frac{n}{n+7}I_{n-1}$ if $I_n = \int_0^1 x^n(x-1)^6\,dx$

soln: $I_n = \int_0^1 x^n(x-1)^6\,dx$ let $u = x^n$ $v' = (x-1)^6$

$u' = nx^{n-1}$ $v = \frac{(x-1)^7}{7}$

$\therefore I_n = \left[x^n\cdot\frac{(x-1)^7}{7}\right]_0^1 - \int_0^1 nx^{n-1}\cdot\frac{(x-1)^7}{7}\,dx$

$I_n = 0 - \frac{n}{7}\int_0^1 x^{n-1}(x-1)^7\,dx$ (1)

$I_n = -\frac{n}{7}\int_0^1 x^{n-1}(x-1)(x-1)^6\,dx$

$I_n = -\frac{n}{7}\int_0^1 x^n(x-1)^6 - x^{n-1}(x-1)^6\,dx$ (1)

$I_n = -\frac{n}{7}I_n + \frac{n}{7}\int x^{n-1}(x-1)^6\,dx$

$\therefore I_n + \frac{n}{7}I_n = +\frac{n}{7}\int_0^1 +x^{n-1}(x-1)^6\,dx$

$\therefore I_n\left(1+\frac{n}{7}\right) = +\frac{n}{7}\underbrace{\int_0^1 x^{n-1}(x-1)^6\,dx}_{I_{n-1}}$ (1)

$\therefore I_n\left(\frac{7+n}{7}\right) = \frac{n}{7}I_{n-1}$

$I_n = \frac{n}{7}\times\frac{7}{7+n}I_{n-1}$ $\therefore I_n = \frac{n}{7+n}I_{n-1}$

$\therefore$ shown

Question 3b i)

3 - correct solns.

2 - obtains $I_n = -\frac{n}{7}\int_0^1 x^{n-1}(x-1)^7\,dx$

and significant progress towards solns.

1 - Attempts to use integration by parts, or equivalent merit.

3b ii) $I_3 = \frac{3}{7+3} I_{3-1} = \frac{3}{10} I_2 = \frac{3}{10} \times \frac{2}{9} I_1$

$\therefore I_3 = \frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} I_0$ (1)

$I_3 = \frac{1}{120}\int_0^1 x^0 (x-1)^6\,dx$

$\therefore I_3 = \frac{1}{120}\left[\frac{(x-1)^7}{7}\right]_0^1 = \frac{1}{120}\left[0 - \frac{-1}{7}\right]$

$I_3 = \frac{1}{840}$ (1)

2 marks - correct soln.

1 mark - correctly applies the recurrence rule

Question 1

a) i) $\displaystyle\int \frac{dx}{\sqrt{x(x-4)}} = \int \frac{dx}{\sqrt{(x-2)^2-4}}$ 1

$= \ln\left|(x-2)+\sqrt{(x-2)^2-4}\right| + C$

$= \ln\left|(x-2)+\sqrt{x^2-4}\right| + C$ 1

ii) Let $u = \sin^{-1}x$

$\displaystyle\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\displaystyle\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx = \int e^u\,du$ 1

$= e^{\sin^{-1}x} + C$ 1

iii) $\displaystyle\int \frac{1}{1+e^x}\,dx = \int 1 - \frac{e^x}{1+e^x}\,dx$ 1

$= x - \ln(1+e^x) + C$ 1

b) Let $t = \tan\frac{x}{2}$

$\displaystyle\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

$= \frac{1}{2}(1+t^2)$

$\therefore\ dx = \dfrac{2\,dt}{1+t^2}$

$x=0,\ t=0$

$x=\pi/2,\ t=1$

$\displaystyle\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{2-\sin x+2\cos x} = \int_0^1 \frac{2\,dt}{(1+t^2)\left(2-\frac{2t}{1+t^2}+\frac{2(1-t^2)}{1+t^2}\right)}$ 1

$\displaystyle= \int_0^1 \frac{1}{2-t}\,dt$

$= \left[-\ln(2-t)\right]_0^1$ 1

$= -\ln 1 + \ln 2$

$= \ln 2$ 1

Question 2

a (i) $3x^2 - 4x + 3 \equiv A(x^2 - x + 2) + (Bx + C)(x - 1)$

sub. $x = 1$, $3 - 4 + 3 = A(1 - 1 + 2)$

$A = 1$

2 marks for correct answers

1 mark for finding two correct values.

Equating coefficients of $x^2$

$A + B = 3$

$1 + B = 3$

$B = 2$

Constant term

$2A - C = 3$

$2 - C = 3$

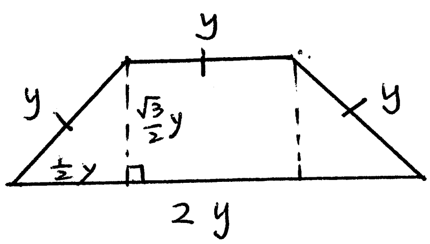
$C = -1$

(ii) $I = \int \frac{1}{x-1}\,dx + \int \frac{2x-1}{x^2-x+2}\,dx$

$= \ln(x-1) + \ln(x^2-x+2) + C$

1 1

(b) (i)




$A = \frac{1}{2}(y + 2y)\frac{\sqrt{3}}{2}y$

$= \frac{3\sqrt{3}\,y^2}{4}$

$= \frac{3\sqrt{3}}{4}\ln(x+1)$ 1

$\Delta V = A\,\Delta x$

$= \frac{3\sqrt{3}}{4}\ln(x+1)\,\Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=0}^{3} \frac{3\sqrt{3}}{4}\ln(x+1)\,\Delta x$

$= \frac{3\sqrt{3}}{4}\int_0^3 \ln(x+1)\,dx$ 1

ii) Let $u = x+1$

$$\frac{du}{dx} = 1$$

$x = 0,\ u = 1$

$x = 3 \quad u = 4$

$$\therefore V = \frac{3\sqrt{3}}{4}\int_1^4 \ln u \, du \qquad \frac{1}{}$$

$$= \frac{3\sqrt{3}}{4}\left\{ [u\ln u]_1^4 - \int_1^4 1\,du \right\} \qquad \frac{1}{}$$

$$= \frac{3\sqrt{3}}{4}\left\{ (4\ln 4 - 4) - (0-1) \right\}$$

$$= \frac{3\sqrt{3}}{4}(4\ln 4 - 3) \qquad \frac{1}{}$$

$$= \frac{3\sqrt{3}}{4}(8\ln 2 - 3) \text{ cubic units}$$

Question 3

(a)
$$\int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin^3(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin^3 x}{1+\cos^2 x}\,dx$$

$$= \int_0^{\pi} \frac{\pi\sin^3 x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx$$

$$2\int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{\pi\sin^3 x}{1+\cos^2 x}\,dx \qquad \frac{1}{}$$

$$\int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin^3 x\,dx}{1+\cos^2 x}$$

Let $u = \cos x$

$du = -\sin x\,dx$

$x = 0,\ u = 1$

$x = 1 \quad u = -1$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin^2 x}{1+\cos^2 x}\sin x\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{1-\cos^2 x}{1+\cos^2 x}\sin x\,dx$$

$$= \frac{\pi}{2}(-1)\int_1^{-1} \frac{1-u^2}{1+u^2}\,du \qquad 1$$

$$= \frac{\pi}{2}\times 2\int_0^1 \frac{2-(1+u^2)}{1+u^2}\,du$$

$$= \pi\int_0^1 \frac{2}{1+u^2} - 1\,du$$

$$= \pi\left[2\tan^{-1}u - u\right]_0^1 \qquad 1$$

$$= \pi\left(2\times\frac{\pi}{4} - 1\right)$$

$$= \frac{\pi}{2}(\pi - 2) \qquad 1$$

(b) Let $x = 2\tan\theta$

$\frac{dx}{d\theta} = 2\sec^2\theta$

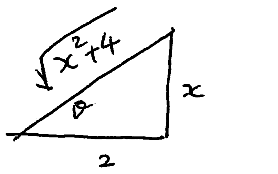
$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4}$

$= 2\sqrt{\tan^2\theta + 1}$ 1

$= 2\sec\theta$

$\therefore \int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{4\tan^2\theta \cdot 2\sec\theta}\, d\theta$

$= \int \frac{\sec\theta}{4\tan^2\theta}\, d\theta$




$= \frac{1}{4}\int \frac{1}{\cos\theta \times \frac{\sin^2\theta}{\cos^2\theta}}\, d\theta$

$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$ 1

$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

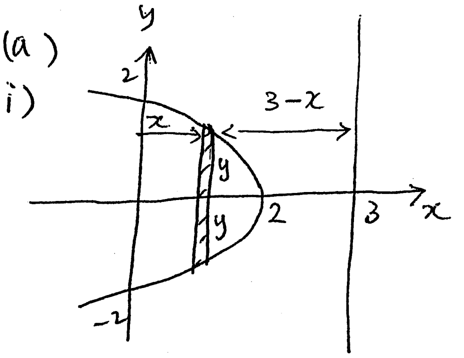
$= -\frac{1}{4} \times \frac{1}{\sin\theta} + C$ 1

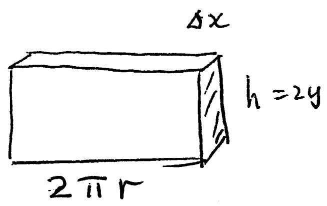
$= -\frac{\sqrt{x^2+4}}{4x} + C$ 1

Question 4

(a)

i)




$r = 3 - x$

$\Delta V = 2\pi r h\, \Delta x$

$= 2\pi(3-x) \times 2y\, \Delta x$

$= 4\pi(3-x)\sqrt{2-x}\, \Delta x$ 1

ii) $V = \lim_{\Delta x \to 0} \sum_{x=0}^{2} 4\pi \cdot (3-x)\sqrt{2-x}\ \Delta x$

$= 4\pi \int_0^2 (3-x)\sqrt{2-x}\, dx$ 1

$= -4\pi \int_2^0 (1+u)\sqrt{u}\, du$ 1

Let $u = 2-x$
$du = -dx$
$x=0,\ u=2$
$x=2,\ u=0$

$= 4\pi \int_0^2 \sqrt{u} + u\sqrt{u}\, du$

$= 4\pi \left[ \frac{2}{3} u^{3/2} + \frac{2}{5} u^{5/2} \right]_0^2$ 1

$= 8\pi \left[ u^{3/2} \left( \frac{1}{3} + \frac{u}{5} \right) \right]_0^2$

$= 8\pi \times 2\sqrt{2} \left( \frac{1}{3} + \frac{2}{5} \right)$

$= \frac{176\sqrt{2}}{15} \pi$ cubic units 1

(b)

i) $\int_0^1 x^n \sqrt{1-x^2}\, dx = \int_0^1 x^{n-1} x\sqrt{1-x^2}\, dx$

$= \left[ -\frac{1}{3} x^{n-1} (1-x^2)^{3/2} \right]_0^1 + \frac{n-1}{3} \int_0^1 x^{n-2} (1-x^2)\sqrt{1-x^2}\, dx$ 1

$= 0 + \left(\frac{n-1}{3}\right) \int_0^1 \left\{ x^{n-2}\sqrt{1-x^2} - x^n\sqrt{1-x^2} \right\} dx$ 1

$= -\left(\frac{n-1}{3}\right) I_n + \left(\frac{n-1}{3}\right) \cdot I_{n-2}$

$\left(1 + \frac{n-1}{3}\right) I_n = \frac{(n-1)}{3} I_{n-2}$

$I_n = \frac{n-1}{n+2} I_{n-2}$ 1

ii) $I_0 = \int_0^1 \sqrt{1-x^2}\, dx$

$= \frac{\pi}{4}$ 1 ( $\frac{1}{4}$ of a circle, radius 1)

$I_2 = \frac{1}{4} I_0$
$= \frac{\pi}{16}$

$I_4 = \frac{3}{6} I_2$
$= \frac{\pi}{32}$ 1