BAULKHAM HILLS HIGH SCHOOL

2017
YEAR 12
TASK 3 JUNE

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black or blue pen
- Black pen is preferred
- Board-approved calculators may be used
- All relevant mathematical reasoning and/or calculations must be shown
- A standard integral sheet is provided at the back of this paper.

Total marks – 39
(Pages 2-5)

Questions 1 to 3
(13 marks each)

QUESTION 1 (13 marks) Start on the appropriate page of your answer booklet

(a) Find $\displaystyle\int \frac{1}{x^2 + 2x + 5}\,dx$ **2**

(b) Find the value of the constants A and B such that

(i) $\displaystyle\frac{1}{u^2 + 3u + 2} = \frac{A}{1+u} + \frac{B}{2+u}$ **2**

(ii) Hence deduce the exact value of the integral $\displaystyle\int_0^1 \frac{e^t}{e^{2t} + 3e^t + 2}\,\mathrm{dt}$ **3**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2 + 3\sin x + 2\cos x}$ using the substitution $t = \tan\frac{x}{2}$. **3**

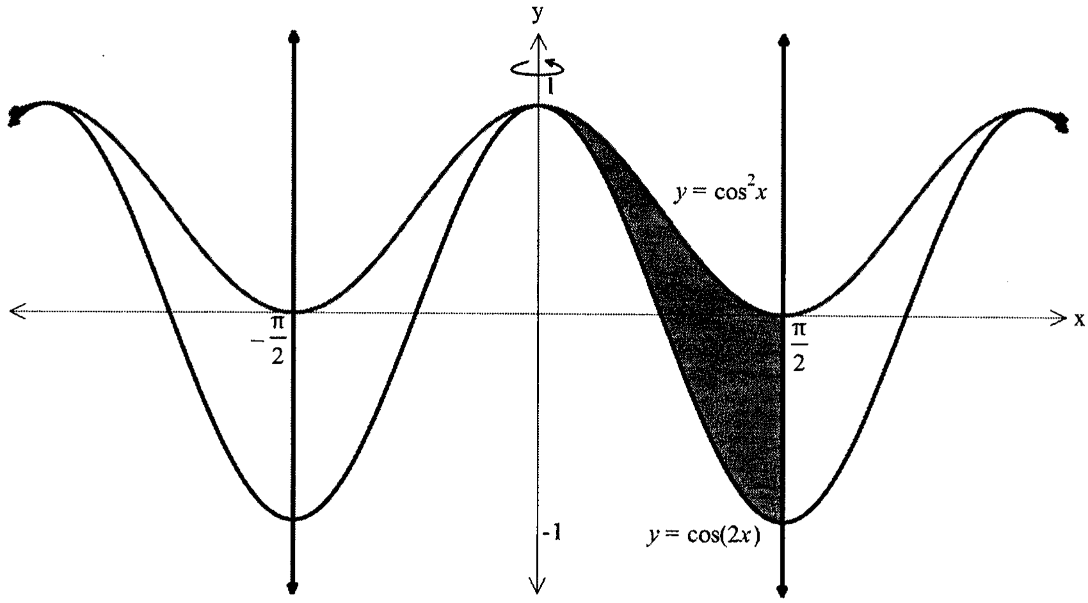
(d) By using the fact that $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$ evaluate $\displaystyle\int_0^{\pi} \frac{x\sin x\,dx}{1 + \cos^2 x}$. **3**

End of Question 1

QUESTION 2 (13 marks) Start on the appropriate page of your answer booklet.

(a) Find $\int \frac{x}{\sqrt{16-x^2}}\,dx$ **2**

(b) The area between the graphs of $y = \cos^2 x$ and $y = \cos 2x$ for $0 \le x \le \frac{\pi}{2}$ and $x = \frac{\pi}{2}$ is rotated about the y axis.



By considering cylindrical shells, find the volume of the solid formed, in terms of π. **4**

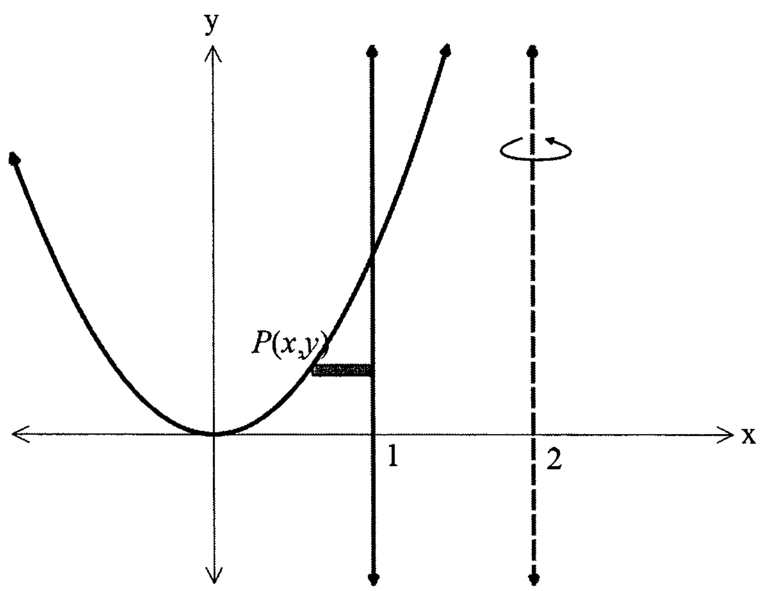
(c) Use the properties of odd and even functions to evaluate $\int_{-4}^{4} \cos x(e^x - e^{-x})\,dx$. **3**

Justify your answer with reasoning.

Question 2 continues on the next page

(d) The area bounded by $x = 1, y = 0$ and $y = x^2$ is rotated about the line $x = 2$.

The volume of the solid formed is to be determined by taking slices perpendicular to the axis of rotation.



(i) Show that the area of the annulus for a slice is $A = \pi(3 - 4\sqrt{y} + y)$ **1**

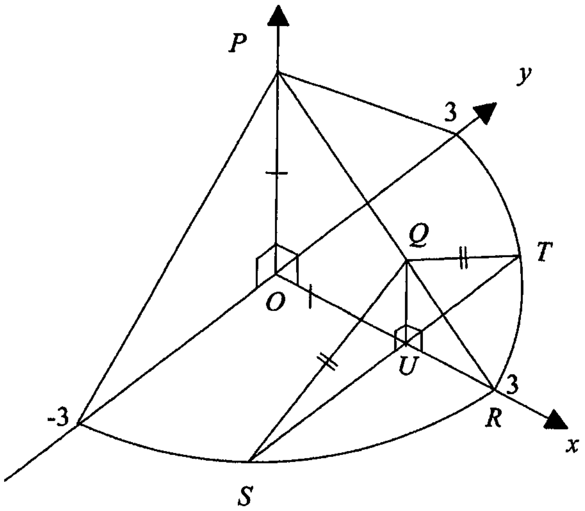
(ii) Find the volume of the solid formed. **3**

End of Question 2

QUESTION 3 (13 marks) Start on the appropriate page of your answer booklet.

(a) Find $\int x^3 \sin x^2 dx$ **3**

(b) A solid figure has a semi circular base of radius $3cm$. Cross sections taken perpendicular to the x axis are isosceles triangles.



The vertical cross section containing the radius OR of the base of the solid is an isosceles right angled triangle ORP where $OR=OP$.

(i) Show that the area of the triangle SQT is given by **2**

$A = (3-x)(9-x^2)^{\frac{1}{2}}$ where $x = OU$.

(ii) Show that the volume of the solid is $\frac{1}{4}(27\pi - 36)$ cm^3. **3**

(c) Let $I_n = \int \frac{x^n}{1+x^2}\, dx$ for $n \geq 0$.

(i) Show that $I_n = \frac{x^{n-1}}{n-1} - I_{n-2}$ **2**

(ii) Hence evaluate $\int_1^3 \frac{x^5}{1+x^2}\, dx$. **3**

End of examination

EXTENSION 2 - JUNE 2017

1 a) $\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4}$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$

② correct answer
① completes the square

b)(i) $\frac{1}{u^2+3u+2} = \frac{A}{1+u} + \frac{B}{2+u}$

$1 = A(2+u)+B(1+u)$

let $u=-2$, $1 = 0-B$

$B=-1$

let $u=-1$, $1 = A+0$

$\therefore A=1$ and $B=-1$

② correct values
① A or B correct

(ii) $\int_0^1 \frac{e^t\,dt}{e^{2t}+3e^t+2}$ let $u=e^t$

$du = e^t\,dt$

when $t=1$, $u=e$

$t=0$, $u=1$

$\int_1^e \frac{du}{u^2+3u+2}$

$= \int_1^e \frac{du}{u+1} - \int_1^e \frac{du}{u+2}$

$= \left[\ln(u+1) - \ln(u+2)\right]_1^e$

$= \left[\ln\left(\frac{u+1}{u+2}\right)\right]_1^e$

$= \ln\left(\frac{e+1}{e+2}\right) - \ln\left(\frac{2}{3}\right)$

$= \ln\frac{3(e+1)}{2(e+2)}$

accept either

③ correct answer
② converts to integral in u and integrates
② converts to integral in u, incorrectly integrates and evaluates
① converts to integral in u.

1 (c) $\int_0^{\frac{\pi}{2}} \frac{dx}{2+3\sin x+2\cos x}$ $t=\tan\frac{x}{2}$

$\frac{2dt}{1+t^2} = dx$

when $x=\frac{\pi}{2}$, $t=\tan\frac{\pi}{4}=1$

$x=0$, $t=\tan 0 = 0$

$\int_0^1 \frac{\frac{2dt}{1+t^2}}{2+3\frac{2t}{1+t^2}+2\left(\frac{1-t^2}{1+t^2}\right)}$

$= \int_0^1 \frac{2dt}{2+2t^2+6t+2-2t^2}$

$= \int_0^1 \frac{2dt}{6t+4}$

$= \int_0^1 \frac{dt}{3t+2}$

$= \frac{1}{3}\left[\ln(3t+2)\right]_0^1$

$= \frac{1}{3}\ln 5 - \ln 2$

$= \frac{1}{3}\ln\left(\frac{5}{2}\right)$

③ correct answer
② expresses as integral in t and correctly integrates
② expresses as integral in t and evaluates integral with incorrect limits
① expresses as integral in t and integrates with wrong limits.

1 (d) next page.

1 (d) $\displaystyle\int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x}$

$$= \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx \quad ✓$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin x\,dx}{1+[-\cos x]^2}$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin x\,dx}{1+\cos^2 x}$$

So that $\displaystyle\int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x} = \int_0^{\pi}\left(\frac{\pi\sin x}{1+\cos^2 x} - \frac{x\sin x}{1+\cos^2 x}\right)dx$

$$2\int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x} = \pi\int_0^{\pi}\frac{\sin x\,dx}{1+\cos^2 x} \quad ✓$$

$$= -\pi\left[\tan^{-1}(\cos x)\right]_0^{\pi}$$

$$= -\pi\left(\tan^{-1}(\cos\pi) - \tan^{-1}(\cos 0)\right) \quad ✓$$

$$= -\pi\left(\tan^{-1} -1 - \tan^{-1} 1\right)$$

$$= -\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right)$$

$$2\int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x} = \frac{\pi^2}{2}$$

$$\int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x} = \frac{\pi^2}{4} \quad ✓$$

(3) correct answer

(2) applies formula and evaluates $[\tan^{-1}(\cos x)]_0^{\pi}$

(2) applies formula and obtains expression for $2\int_0^{\pi}\frac{x\sin x\,dx}{1+\cos^2 x}$.

(1) applies formula

2 (a) $\displaystyle\int \frac{x\,dx}{\sqrt{16-x^2}}$

$$= -\frac{1}{2}\int -2x\,(16-x^2)^{-\frac{1}{2}}\,dx$$

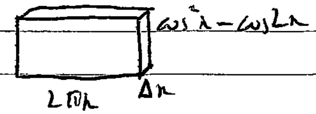
$$= -\frac{1}{2}\,\frac{(16-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= -\sqrt{16-x^2} + C$$

(2) correct answer

(1) obtains $a\sqrt{16-x^2}$ where $a \neq -1$. (or equivalent merit)

2 (b)



Volume of shell $\Delta V = 2\pi x(\cos^2 x - \cos 2x)\Delta x$

$$= \pi x(2\cos^2 x - 2\cos 2x)\,\Delta x$$

$$= \pi x(\cos 2x + 1 - 2\cos 2x)\,\Delta x$$

$$= \pi x(1-\cos 2x)\,\Delta x$$

$$V = \lim_{\Delta x\to 0}\sum_{x=0}^{\pi/2} \pi(x - x\cos 2x)\,\Delta x$$

$$= \pi\int_0^{\pi/2} x - x\cos 2x\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^{\pi/2} - \pi\int_0^{\pi/2} x\cos 2x\,dx$$

$$= \pi\,\frac{\pi^2}{8} - \pi\left[\left[\frac{x\sin 2x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\sin 2x}{2}\,dx\right]$$

$$= \frac{\pi^3}{8} - \pi\left(\frac{\pi}{4}\sin 0 - 0\right) + \frac{\pi}{2}\int_0^{\pi/2}\sin 2x\,dx$$

$$= \frac{\pi^3}{8} - \frac{\pi}{4}\left[\cos 2x\right]_0^{\pi/2}$$

$$= \frac{\pi^3}{8} - \frac{\pi}{4}(\cos\pi - \cos 0)$$

$$= \frac{\pi^3}{8} - \frac{\pi}{4}(-1-1)$$

$$= \frac{\pi^3}{8} + \frac{\pi}{2}$$

(4) correct answer

(3) integrates by parts $\pi\int x - x\cos 2x\,dx$

(2) expresses volume integral as $\pi\int x - x\cos 2x\,dx$

(1) finds volume of a slice

2 (c) $\int_{-4}^{4} \cos x\left(e^{x}-e^{-x}\right)dx$

Let $f(x) = \cos x\left(e^{x}-e^{-x}\right)$

$f(-x) = \cos(-x)\left(e^{-x}-e^{x}\right)$

$f(-x) = -\cos x\left(e^{x}-e^{-x}\right)$ (cos x is even)

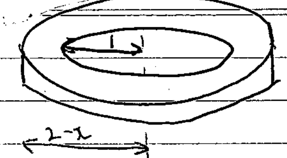
$= -f(x)$

$\therefore f(x)$ is odd

$\int_{-4}^{4} \cos x\left(e^{x}-e^{-x}\right)dx = 0$

(3) correct value with reason
(2) states function is odd and gives value as 0.
(2) shows function is odd
(1) gives answer as 0 without reasons.

(d)



$x=2$

(i) $A = \pi\left[(2-x)^2 - 1^2\right]$

$= \pi\left(4-4x+x^2-1\right)$

$A = \pi\left(3-4x+x^2\right)$

(1) correct proof

(ii) Volume of a slice,

$\Delta V = \pi\left(3-4x+x^2\right)\Delta y$

$= \pi\left(3-4\sqrt{y}+y\right)\Delta y$

Total volume $= \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi\left(3-4\sqrt{y}+y\right)\Delta y$

$= \pi\int_0^1 3-4y^{1/2}+y\,dy$

$= \pi\left[3y - \frac{4y^{3/2}}{\frac{3}{2}} + \frac{y^2}{2}\right]_0^1$

$= \pi\left(3-\frac{8}{3}+\frac{1}{2}\right) - (0-0+0)$

Volume $= \frac{5\pi}{6}$ units^3

(3) correct solution
(2) expresses as correct integral including having expressed as a limit
(2) correct answer without having expressed as a limit
(1) finds volume of a slice in terms of y

3 (a) $\int x^3 \sin x^2\,dx$

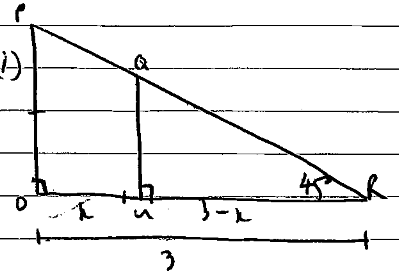
$= \int x^2 \cdot x\sin x^2\,dx$ $\quad u = x^2 \quad v' = x\sin x^2$; $\quad u' = 2x \quad v = -\frac{1}{2}\cos x^2$

$= \frac{-x^2}{2}\cos x^2 + \int x\cos x^2\,dx$

$= \frac{-x^2}{2}\cos x^2 + \frac{1}{2}\sin x^2 + C$

(3) correct answer
(2) correctly splits integral and uses integration by parts
(1) correctly splits and attempts to integrate by parts

3 (b)(i)



$\tan 45° = \frac{QU}{UR}$

$QU = UR$

$= 3-x$

$\therefore$ Area of $\Delta QST = \frac{2y(3-x)}{2}$

$= y(3-x)$

$= \sqrt{9-x^2}\,(3-x)$

$\therefore A = (3-x)(9-x^2)^{1/2}$

(3) correct solution
(1) finds QU = 3−x
(1) expresses ST as $2\sqrt{9-x^2}$
(1) attempts to use area of triangle.

(ii) Volume of slice, $\Delta V = (3-x)(9-x^2)^{1/2}\Delta x$

Volume, $V = \lim_{\Delta x \to 0} \sum_{x=0}^{3} (3-x)(9-x^2)^{1/2}\Delta x$

$= \int_0^3 (3-x)(9-x^2)^{1/2}dx$

$= 3\int_0^3 \sqrt{9-x^2}\,dx + \int_0^3 -x(9-x^2)^{1/2}dx$

$= 3 \times \frac{\pi \times 3^2}{4} + \frac{1}{2}\int_0^3 -2x(9-x^2)^{1/2}dx$

$= \frac{27\pi}{4} + \frac{1}{2}\left[\frac{2(9-x^2)^{3/2}}{3}\right]_0^3$

$= \frac{27\pi}{4} + \frac{1}{3}(9-x^2)^{3/2}$

(3) correct answer
(2) splits integral and evaluates one correctly.
(1) evaluates answer without showing desired result (without expressing as a limit.)
(1) expresses volume as a limit

$$= \frac{27\pi}{4} + \frac{1}{3}\left(0 - 9^{3/2}\right)$$

$$= \frac{27\pi - 9}{4}$$

$$= \frac{1}{4}\left(27\pi - 36\right) \text{ cm}^3 \text{ as req'd.}$$

(c) Let $I_n = \int \frac{x^n}{1+x^2}\,dx$

(i) $I_n = \int \frac{x^{n-2}\, x^2}{1+x^2}\,dx$

$$= \int x^{n-2} \frac{(1+x^2-1)}{1+x^2}\,dx \checkmark$$

$$= \int x^{n-2}\,dx - \int \frac{x^{n-2}}{1+x^2}\,dx$$

$$I_n = \frac{x^{n-1}}{n-1} - I_{n-2} \checkmark$$

(2) correct solution

(1) splits integral and manipulates algebraically

(ii) $I_5 = \frac{x^4}{4} - I_3$

$$= \frac{x^4}{4} - \left(\frac{x^2}{2} - I_1\right)$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_1^3 + \int_1^3 \frac{x}{1+x^2}\,dx$$

$$= \left(\frac{81}{4} - \frac{9}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) + \frac{1}{2}\left[\ln(1+x^2)\right]_1^3 \checkmark$$

$$= 20 - 4 + \frac{1}{2}(\ln 10 - \ln 2)$$

$$= 16 + \frac{1}{2}\ln 5 \checkmark$$

(3) correct answer

(2) applies recurrence ✓ formula and integrates

(1) expresses I_5 in terms of I_1