GIRRAWEEN HIGH SCHOOL
MATHEMATICS

Year 12 Extension 2 Task 3 | Monday 23rd June 2002

Instructions: a) Write all your answers on your own paper.
b) Show all necessary working.
c) Marks may be deducted for careless or badly arranged work.

Time Allowed: 80 minutes

Question 1 ***(27 marks)*** — ***Marks***

a) Find the following integrals:

(i) $\int \tan^3 x dx$ — *4*

(ii) $\int \frac{dx}{x^3\sqrt{x^2-4}}$ — *4*

(iii) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$ — *4*

(iv) $\int_0^{\frac{\pi}{4}} x\sec^2 x dx$ — *4*

(v) $\int \frac{x+2}{\sqrt{x^2-6x+7}}dx$ — *4*

(vi) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^4 x \sin^5 x dx$ — *3*

(vii) $\int \frac{7x^2-5x+4}{(x-1)(x^2+1)}dx$ — *4*

Question 2 ***(9 marks)***

a) The integral $I_n = \int_0^{\frac{\pi}{4}} x^n \cos 2x dx$;

(i) Prove that $I_n = \frac{1}{2}\left(\frac{\pi}{4}\right)^n - \frac{n(n-1)}{4}I_{n-2}$ — *4*

(ii) Using your answer to (i), evaluate $\int_0^{\frac{\pi}{4}} x^4 \cos 2x dx$ — *3*

b) In your own words explain the error in the following solution: — *2*

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x dx}{\cos x} \\
&= \int \sec x \sin x dx \\
&= -\sec x \cos x + \int \sec x \tan x \cos x dx \\
&= -\sec x \cos x + \int \tan x dx \\
\int \tan x dx - \int \tan x dx &= -\sec x \cos x \\
0 &= -1
\end{aligned}$$

Question 3 *(24 marks)* *Marks*

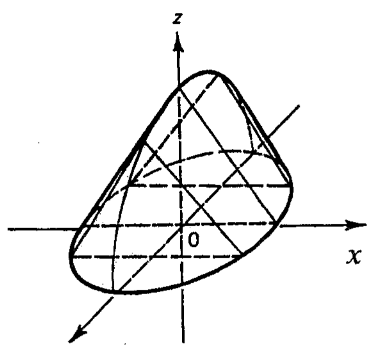
a) Find the volume generated by rotating the area bounded by $y = \log x$, $y = 0$ and $x = 3$, about the x-axis. 5

b) Consider the function $y = 2x - x^2$;

(i) Show that when x is written as the subject, the resulting equation is $x = 1 \pm \sqrt{1-y}$ 2

(ii) By taking slices perpendicular to the y-axis, calculate the volume of the solid generated when the area bounded by $y = 2x - x^2$ and the x-axis is rotated about the y-axis. 5

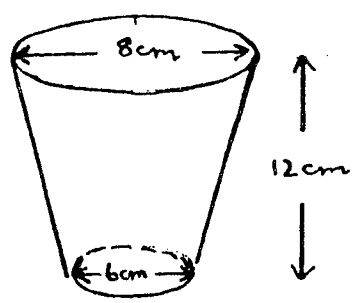
c) The solid shown has a circular base of radius 2 units. Vertical cross sections perpendicular to the y-axis are equilateral triangles.



(i) Show that the area of one of these slices is given by $A(y) = \sqrt{3}(4 - y^2)$ 3

(ii) Hence calculate the volume of the solid. 3

d) A drinking glass is in the shape of a truncated cone in which the internal diameters of the top and the bottom are 8 cm and 6 cm respectively. The internal height of the glass is 12 cm.



(i) Show that the cross section taken x cm above the base has radius $\left(3 + \frac{x}{12}\right)$ cm. 2

(ii) Hence find the volume of fluid the glass can hold. 4

Question 1 (27)

a)

(i) $\int \tan^3 x\,dx$

$= \int \tan x(\sec^2 x - 1)\,dx$

$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$

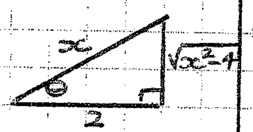
$= \int u\,du + \log\cos x + c$ $\quad u = \tan x$

$= \frac{1}{2}u^2 + \log\cos x + c$ $\quad du = \sec^2 x\,dx$

$= \frac{1}{2}\tan^2 x + \log\cos x + c$ (4)

(ii) $\int \frac{dx}{x^3\sqrt{x^2-4}}$ $\quad x = 2\sec\theta$, $\quad dx = 2\sec\theta\tan\theta\,d\theta$

$= \int \frac{2\sec\theta\tan\theta\,d\theta}{8\sec^3\theta\sqrt{4\sec^2\theta - 4}}$



$= \frac{1}{8}\int \frac{\sec\theta\tan\theta\,d\theta}{\sec^3\theta\tan\theta}$

$= \frac{1}{8}\int \cos^2\theta\,d\theta$

$= \frac{1}{16}\int (1+\cos 2\theta)\,d\theta$

$= \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + c$

$= \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + c$

$= \frac{1}{16}\sec^{-1}\frac{x}{2} + \frac{1}{16}\cdot\frac{\sqrt{x^2-4}}{x}\cdot\frac{2}{x} + c$

$= \frac{1}{16}\sec^{-1}\frac{x}{2} + \frac{\sqrt{x^2-4}}{8x^2} + c$ (4)

(iii) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$ $\quad t = \tan\frac{x}{2}$, $\quad dx = \frac{2dt}{1+t^2}$

$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}$

$= \int_0^1 \frac{2dt}{1+t^2+2t+1-t^2}$

$= \int_0^1 \frac{2dt}{2+2t}$

$= \int_0^1 \frac{dt}{1+t}$

$= \left[\log(1+t)\right]_0^1$

$= \log 2 - \log 1$

$= \log 2$ (4)

(iv) $\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$ $\quad u = x \quad v = \tan x$, $\quad du = dx \quad dv = \sec^2 x\,dx$

$= \left[x\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,dx$

$= \left[x\tan x + \log\cos x\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4}\tan\frac{\pi}{4} + \log\cos\frac{\pi}{4} - 0 + \log\cos 0$

$= \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} + \log 1$

$= \frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right)$ (4)

(vi) $\int \frac{x+2}{\sqrt{x^2-6x+7}}\,dx$ $\quad u = x^2-6x+7$, $\quad du = (2x-6)dx$

$= \frac{1}{2}\int \frac{2x-6}{\sqrt{x^2-6x+7}}\,dx + 5\int \frac{dx}{\sqrt{(x-3)^2-2}}$

$= \frac{1}{2}\int u^{-\frac{1}{2}}\,du + 5\log(x-3+\sqrt{x^2-6x+7}$

$= u^{\frac{1}{2}} + 5\log(x-3+\sqrt{x^2-6x+7}) +$

$= \sqrt{x^2-6x+7} + 5\log(x-3+\sqrt{x^2-6x+7})$

(vii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x\sin^5 x\,dx$

$\cos^4 x$ = even function

$\sin^5 x$ = odd function

$\therefore \cos^4 x\sin^5 x$ = odd function

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x\sin^5 x\,dx = 0$ (3)

(viii) $\int \frac{7x^2-5x+4}{(x-1)(x^2+1)}\,dx$

$A(x^2+1) + (Bx+C)(x-1) = 7x^2-5x+4$

$x = 1$: $\quad 2A = 6$, $\quad A = 3$

$x = i$: $\quad (Bi+C)(i-1) = -7-5i+4$

$(-B-C) + i(C-B) = -3-5i$

$B + C = 3$

$B - C = 5$

$2B = 8$

$B = 4 \quad \therefore C = -1$

$\int \frac{7x^2-5x+4}{(x-1)(x^2+1)}\,dx$

$= \int \left[\frac{3}{x-1} + \frac{4x}{x^2+1} - \frac{1}{x^2+1}\right]dx$

$= 3\log(x-1) + 2\log(x^2+1) - \tan^{-1}x + c$ (4)

Question 2 (9)

a) $I_n = \int_0^{\frac{\pi}{4}} x^n \cos 2x\,dx$ $\quad u = x^n \quad v = \frac{1}{2}\sin 2x$
$\quad du = nx^{n-1}dx \quad dv = \cos 2x\,dx$

$= \left[\frac{1}{2}x^n \sin 2x\right]_0^{\frac{\pi}{4}} - \frac{1}{2}n\int_0^{\frac{\pi}{4}} x^{n-1}\sin 2x\,dx$

$u = x^{n-1} \quad v = -\frac{1}{2}\cos 2x$
$du = (n-1)x^{n-2}dx \quad dv = \sin 2x\,dx$

$= \left[\frac{1}{2}x^n \sin 2x + \frac{1}{4}nx^{n-1}\cos 2x\right]_0^{\frac{\pi}{4}} - \frac{1}{4}n(n-1)\int_0^{\frac{\pi}{4}} x^{n-2}\cos 2x\,dx$

$= \frac{1}{2}\left(\frac{\pi}{4}\right)^n \sin\pi + \frac{1}{4}n\left(\frac{\pi}{4}\right)^{n-1}\cos\pi - 0 - \frac{n(n-1)}{4}I_{n-2}$

$= \frac{1}{2}\left(\frac{\pi}{4}\right)^n - \frac{n(n-1)}{4}I_{n-2}$ (4)

(ii) $\int_0^{\frac{\pi}{4}} x^4 \cos 2x\,dx$

$= \frac{1}{2}\left(\frac{\pi}{4}\right)^4 - \frac{4(3)}{4}I_2$

$= \frac{\pi^4}{512} - \frac{3}{2}\left(\frac{\pi}{4}\right)^2 + \frac{2(1)}{4}I_0$

$= \frac{\pi^4}{512} - \frac{3\pi^2}{32} + \frac{1}{2}\int_0^{\frac{\pi}{4}}\cos 2x\,dx$

$= \frac{\pi^4}{512} - \frac{3\pi^2}{32} + \frac{1}{4}\left[\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi^4}{512} - \frac{3\pi^2}{32} + \frac{1}{4}\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= \frac{\pi^4}{512} - \frac{3\pi^2}{32} + \frac{1}{4}$ (3)

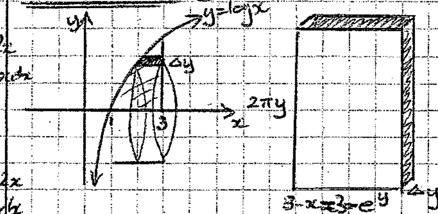
b) $\int \tan x = -\sec x\cos x + c + \int \tan x\,dx$ (ii)

$\int \tan x - \int \tan x\,dx = -\sec x \cos x + c$

$0 = -1 + c$

ie the error is neglecting the constant of integration (2)

Question 3 (24)



$A(y) = 2\pi y(3 - e^y)$

$\Delta V = (6\pi y - 2\pi y e^y)\Delta y$

$V = \lim_{\Delta y \to 0}\sum_{y=0}^{\log 3}(6\pi y - 2\pi y e^y)\Delta y$

$= 2\pi\int_0^{\log 3}(3y - ye^y)\,dy \quad u = y \quad v = e^y \quad du = dy \quad dv = e^y dy$

$= 2\pi\left[\frac{3}{2}y^2 - ye^y\right]_0^{\log 3} + 2\pi\int_0^{\log 3} e^y\,dy$

$= 2\pi\left[\frac{3}{2}y^2 - ye^y + e^y\right]_0^{\log 3}$

$= 2\pi\left[\frac{3}{2}(\log 3)^2 - (\log 3)3 + 3 - 1\right]$

$= \pi\left(3(\log 3)^2 - 6\log 3 + 4\right)$ units^3

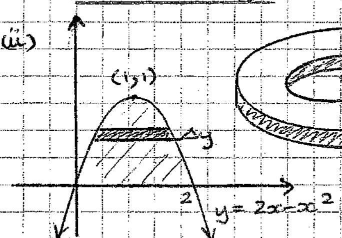
b) $y = 2x - x^2$

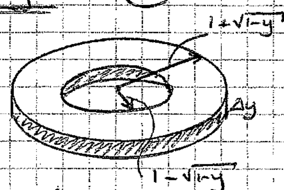
$x^2 - 2x + y = 0$

$x = \frac{2 \pm \sqrt{4 - 4y}}{2}$

$x = \frac{2 \pm 2\sqrt{1-y}}{2}$

$x = 1 \pm \sqrt{1-y}$ (2)





$A(y) = \pi(1 + \sqrt{1-y})^2 - \pi(1 - \sqrt{1-y})^2$

$= \pi\left(1 + 2\sqrt{1-y} + (1-y)^2 - 1 + 2\sqrt{1-y} - (1-y)^2\right)$

$= 4\pi\sqrt{1-y}$

$\Delta V = 4\pi\sqrt{1-y}\,\Delta y$

$V = \lim_{\Delta y \to 0}\sum_{y=0}^{1} 4\pi\sqrt{1-y}\,\Delta y$

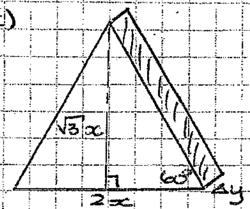
$= 4\pi\int_0^1 (1-y)^{\frac{1}{2}}\,dy$

$= 4\pi\left[-\frac{2}{3}(1-y)^{\frac{3}{2}}\right]_0^1$

$= -\frac{8\pi}{3}(0 - 1)$

$= \frac{8\pi}{3}$ units^3 (5)

c)



$\frac{h}{x} = \tan 60°$

$h = x\tan 60°$

$h = \sqrt{3}x$

$A(y) = \frac{1}{2}(2x)(\sqrt{3}x)$

$= \sqrt{3}x^2$

$= \sqrt{3}(4 - y^2)$ (3)

(ii) $\Delta V = \sqrt{3}(4 - y^2)\Delta y$

$V = \lim_{\Delta y \to 0}\sum_{y=-2}^{2}\sqrt{3}(4 - y^2)\Delta y$

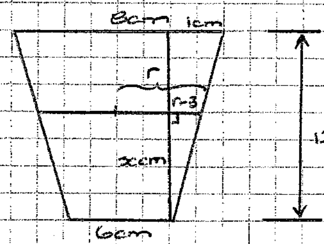
$= 2\sqrt{3}\int_0^2 (4 - y^2)\,dy$

$= 2\sqrt{3}\left[4y - \frac{1}{3}y^3\right]_0^2$

$= 2\sqrt{3}\left(8 - \frac{8}{3} - 0\right)$

$= \frac{32\sqrt{3}}{3}$ units^3 (3)

d)

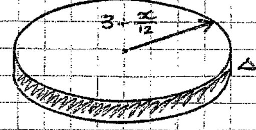


$\frac{r-3}{1} = \frac{x}{12}$

$r - 3 = \frac{x}{12}$

$r = 3 + \frac{x}{12}$ (2)

(ii)



$A(x) = \pi\left(3 + \frac{x}{12}\right)^2$

$= \pi\left(9 + \frac{x}{2} + \frac{x^2}{144}\right)$

$\Delta V = \pi\left(9 + \frac{1}{2}x + \frac{1}{144}x^2\right)\Delta x$

$V = \lim_{\Delta x \to 0}\sum_{x=0}^{12}\pi\left(9 + \frac{1}{2}x + \frac{1}{144}x^2\right)\Delta x$

$= \pi\int_0^{12}\left(9 + \frac{1}{2}x + \frac{1}{144}x^2\right)dx$

$= \pi\left[9x + \frac{1}{4}x^2 + \frac{1}{432}x^3\right]_0^{12}$

$= \pi(108 + 36 + 4 - 0)$

$= 148\pi$ cm^3 (4)