GIRRAWEEN HIGH SCHOOL
MATHEMATICS

Year 12 Extension 2 Task 3 ~~4~~ ? Monday 21st June 2004

Instructions: a) Write all your answers on your own paper.
b) Show all necessary working.
c) Marks may be deducted for careless or badly arranged work.

Time Allowed: 90 minutes

Question 1 *(28 marks)* ***Marks***

a) Find the following integrals:

(i) $\int \sin^3 2x dx$ 4

(ii) $\int x^2 e^{2x} dx$ 4

(iii) $\int \frac{x^2 dx}{\sqrt{(1-x^2)^3}}$ 4

(iv) $\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x} dx$ 4

(v) $\int \frac{x}{\sqrt{1-2x-x^2}} dx$ 4

(vi) $\int \sec^3 x dx$ 4

(vii) $\int \frac{2x+3}{(x-2)(x^2+3)} dx$ 4

Question 2 *(11 marks)*

a) (i) If $I_n = \int_1^e x(\log x)^n dx$ for $n \geq 0$ show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ for $n \geq 1$ 3

(ii) Evaluate $\int_1^e x(\log x)^3 dx$ 3

b) (i) Use the substitution $u = \frac{1}{x}$, show that $\int_{0.5}^{1} \frac{\log x}{1+x^2} dx = \int_2^1 \frac{\log u}{1+u^2} du$ 3

(ii) Hence, or otherwise, evaluate $\int_{0.5}^{2} \frac{\log x}{1+x^2} dx$ 2

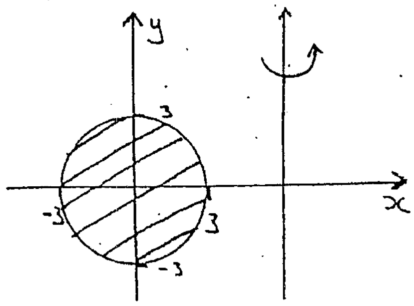
Question 3 *(25 marks)* **Marks**

a) The area bounded by the curve $y = x^2 + 1$ and the line $y = 3 - x$ is rotated about the x-axis.

(i) Sketch the curve and line clearly labelling all the points of intersection. 3

(ii) By considering slices perpendicular to the x-axis, find the volume of the solid formed. 5

b) The diagram shows the region $x^2 + y^2 \leq 9$ and the line $x = 6$



$x = 6$

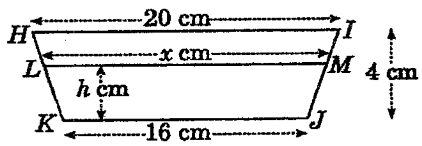
(i) Use the method of cylindrical shells to show that if the region $x^2 + y^2 \leq 9$ is rotated about the line $x = 6$, the volume V of the torus formed is given by; 3

$$V = 4\pi \int_{-3}^{3} (6 - x)\sqrt{9 - x^2}\, dx$$

(ii) Hence find the volume of the torus. 3

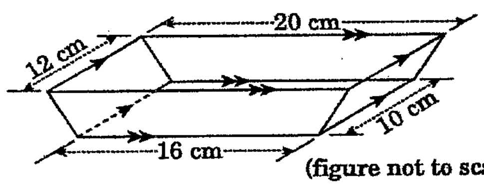
c) A solid has a base in the form of an ellipse with major axis 10 units and minor axis 8 units. If every section perpendicular to the major axis is an isosceles triangle with altitude 6 units, find the volume of the solid. 5

d)



(i) A trapezium $HIJK$ has parallel sides KJ = 16 cm and HI = 20 cm. The distance between these sides is 4 cm. L lies in HK and M lies on IJ such that LM is parallel to KJ. The shortest distance from K to LM is h cm and LM has length x cm. Prove that $x = 16 + h$. 2

(ii)



(figure not to scale) 4

The diagram above is of a cake tin with a rectangular base with sides 16 cm and 10 cm. Its top is also rectangular with dimensions 20 cm and 12 cm. The tin has depth 4 cm and each of its four side faces is a trapezium. Find its volume.

Year 12 Extension 2 – Task 3 2004 Solutions

Question 1 (28)

a) (i) $\int \sin^3 2x\,dx$

$= \int \sin 2x\,(1-\cos^2 2x)\,dx$

$= -\frac{1}{2}\int (1-u^2)\,du$ $\quad u = \cos 2x,\; du = -2\sin 2x\,dx$

$= -\frac{1}{2}(u - \frac{1}{3}u^3) + c$

$= -\frac{1}{2}\cos 2x + \frac{1}{6}\cos^3 2x + c$ (4)

(ii) $\int x^2 e^{2x}\,dx$ $\quad u = x^2,\; v = \frac{1}{2}e^{2x},\; du = 2x\,dx,\; dv = e^{2x}dx$

$= \frac{1}{2}x^2e^{2x} - \int x e^{2x}\,dx$

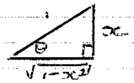
$u = x,\; v = \frac{1}{2}e^{2x},\; du = dx,\; dv = e^{2x}dx$

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\int e^{2x}\,dx$

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$ (4)

(iii) $\int \frac{x^2\,dx}{\sqrt{(1-x^2)^3}}$ $\quad x = \sin\theta,\; dx = \cos\theta\,d\theta$

$= \int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos^3\theta}$



$= \int \tan^2\theta\,d\theta$

$= \int (\sec^2\theta - 1)\,d\theta$

$= \tan\theta - \theta + c$

$= \frac{x}{\sqrt{1-x^2}} - \sin^{-1}x + c$ (4)

(iv) $\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}\,dx$ $\quad t = \tan\frac{x}{2},\; dx = \frac{2dt}{1+t^2}$

$= \int_0^{\sqrt{3}} \frac{\frac{2dt}{1+t^2}}{5 + 4\left(\frac{1-t^2}{1+t^2}\right)}$

$= \int_0^{\sqrt{3}} \frac{2dt}{5+5t^2+4-4t^2}$

$= \int_0^{\sqrt{3}} \frac{2dt}{9+t^2}$

$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^{\sqrt{3}}$

$= \frac{2}{3}\left\{\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0\right\}$

$= \frac{2}{3}\times\frac{\pi}{6}$

$= \frac{\pi}{9}$ (4)

(v) $\int \frac{x\,dx}{\sqrt{1-2x-x^2}}$

$= -\frac{1}{2}\int\frac{-2x-2}{\sqrt{1-2x-x^2}}\,dx - \int\frac{dx}{\sqrt{2-(x+1)^2}}$

$= -\sqrt{1-2x-x^2} - \sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$ (4)

(vi) $\int \sec^3 x = \int \sec x\,\sec^2 x\,dx$

$u = \sec x,\; v = \tan x,\; du = \sec x\tan x\,dx,\; dv = \sec^2 x\,dx$

$= \sec x\tan x - \int \sec x\tan^2 x\,dx$

$= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$= \sec x\tan x - \int \sec^3 x\,dx + \log(\sec x + \tan x) + c$

$\therefore 2\int \sec^3 x\,dx = \sec x\tan x + \log(\sec x + \tan x) + c$

$\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x\tan x + \log(\sec x + \tan x)\right) + c$ (4)

(vii) $\frac{A}{x-2} + \frac{Bx+C}{x^2+3} = \frac{2x+3}{(x-2)(x^2+3)}$

$A(x^2+3) + (Bx+C)(x-2) = 2x+3$

$x=2$: $7A = 7$, $A = 1$ $\quad x=0$: $3A - 2C = 3$, $3 - 2C = 3$, $C = 0$ $\quad x=1$: $4A - B - C = 5$, $4 - B = 5$, $B = -1$

$\int \frac{2x+3}{(x-2)(x^2+3)}\,dx = \int\left[\frac{1}{x-2} - \frac{x}{x^2+3}\right]dx$

$= \log(x-2) - \frac{1}{2}\log(x^2+3) + c$ (4)

Question 2 (11)

a) (i) $I_n = \int_1^e x(\log x)^n\,dx$

$u = (\log x)^n,\; v = \frac{1}{2}x^2,\; du = \frac{n(\log x)^{n-1}}{x}\,dx,\; dv = x\,dx$

$I_n = \left[\frac{1}{2}x^2(\log x)^n\right]_1^e - \frac{n}{2}\int_1^e x(\log x)^{n-1}\,dx$

$= \frac{e^2}{2} - \frac{n}{2}I_{n-1}$ (3)

(ii) $\int_1^e x(\log x)^3\,dx = I_3$

$= \frac{e^2}{2} - \frac{3}{2}I_2$

$= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3}{2}I_1$

$= -\frac{e^2}{4} + \frac{3e^2}{4} - \frac{3}{4}I_0$

$= \frac{e^2}{2} - \frac{3}{4}\int_1^e x\,dx$

$= \frac{e^2}{2} - \frac{3}{4}\left[\frac{1}{2}x^2\right]_1^e$

$= \frac{e^2}{2} - \frac{3}{8}e^2 + \frac{3}{8}$

$= \frac{e^2+3}{8}$ (3)

b) (i) $\int_{0.5}^1 \frac{\log x}{1+x^2}\,dx$ $\quad u = \frac{1}{x} \Rightarrow x = \frac{1}{u},\; dx = -\frac{du}{u^2}$

$= \int_2^1 \frac{-\log\left(\frac{1}{u}\right)}{1+\left(\frac{1}{u}\right)^2}\frac{du}{u^2}$

$= \int_2^1 \frac{\log u}{u^2+1}\,du$ (3)

(ii) $\int_{0.5}^2 \frac{\log x}{1+x^2}\,dx$

$= \int_{0.5}^1 \frac{\log x}{1+x^2}\,dx + \int_1^2 \frac{\log x}{1+x^2}\,dx$

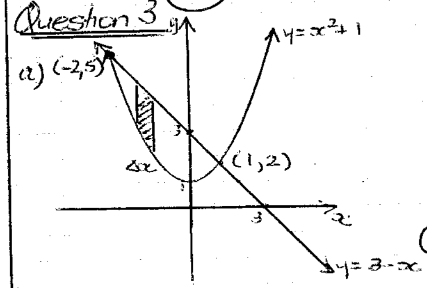
$= \int_2^1 \frac{\log x}{1+x^2}\,dx + \int_1^2 \frac{\log x}{1+x^2}\,dx$

$= -\int_1^2 \frac{\log x}{1+x^2}\,dx + \int_1^2 \frac{\log x}{1+x^2}\,dx$

$= 0$ (2)

Question 3 (25)

a) (i)



(3)

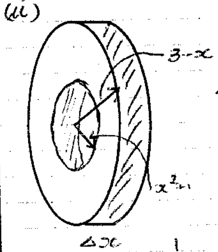
$x^2 + 1 = 3 - x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2$ or $x = 1$

(ii)



$A(x) = \pi\left[(3-x)^2 - (x^2+1)^2\right]$

$\Delta V = \pi(9 - 6x + x^2 - x^4 - 2x^2 - 1)$

$= \pi(8 - 6x - x^2 - x^4)\Delta x$

$V = \lim_{\Delta x \to 0}\sum_{x=-2}^{1}\pi(8 - 6x - x^2 - x^4)\Delta x$

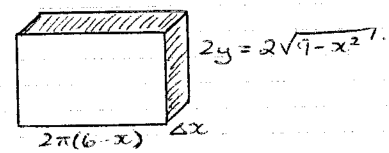
$= \pi\int_{-2}^{1}(8 - 6x - x^2 - x^4)\,dx$

$= \pi\left[8x - 3x^2 - \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{-2}^{1}$

$= \pi\left\{\left(8 - 3 - \frac{1}{3} - \frac{1}{5}\right) - \left(-16 - 12 + \frac{8}{3} + \frac{32}{5}\right)\right\}$

$= \frac{117\pi}{5}$ units³ (5)

b)



$A(x) = 4\pi(6-x)\sqrt{9-x^2}$

$\Delta V = 4\pi(6-x)\sqrt{9-x^2}\,\Delta x$

$V = \lim_{\Delta x \to 0}\sum_{x=-3}^{3}4\pi(6-x)\sqrt{9-x^2}\,\Delta x$

$= 4\pi\int_{-3}^{3}(6-x)\sqrt{9-x^2}\,dx$ (3)

(ii) $V = 4\pi\int_{-3}^{3}(6-x)\sqrt{9-x^2}\,dx$

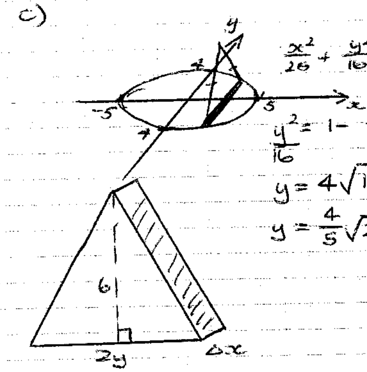
$= 24\pi\int_{-3}^{3}\sqrt{9-x^2}\,dx - 4\pi\int_{-3}^{3}x\sqrt{9-x^2}\,dx$

↑ semicircle ↑ odd x even = odd function

$= 24\pi \times \frac{1}{2}\pi(3)^2 - 0$

$= \underline{108\pi^2 \text{ units}^3}$ (3)

c)



$\frac{x^2}{25} + \frac{y^2}{16} = 1$

$\frac{y^2}{16} = 1 - \frac{x^2}{25}$

$y = 4\sqrt{1-\frac{x^2}{25}}$

$y = \frac{4}{5}\sqrt{25-x^2}$

$= \frac{8}{5}\sqrt{25-x^2}$

$A(x) = \frac{1}{2} \times \frac{8}{5}\sqrt{25-x^2} \times 6$

$= \frac{24}{5}\sqrt{25-x^2}$

$\Delta V = \frac{24}{5}\sqrt{25-x^2}\,\Delta x$

$V = \lim_{\Delta x \to 0}\sum_{x=-5}^{5}\frac{24}{5}\sqrt{25-x^2}\,\Delta x$

$= \frac{24}{5}\int_{-5}^{5}\sqrt{25-x^2}\,dx$

$= \frac{24}{5} \times \frac{1}{2}\pi(5)^2$

$= \underline{60\pi \text{ units}^3}$ (5)

d) $m = \frac{4}{20-16}$

$= 1$

$h - 0 = 1(x - 16)$

$= x - 16$

$\underline{x = h + 16}$ (2)

(ii) y relationship

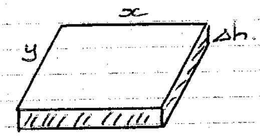
$m = \frac{4}{12-10}$

$= 2.$

$h - 0 = 2(y - 10)$

$h = 2y - 20$

$y = \frac{1}{2}h + 10$



$A(h) = (h+16)(\frac{1}{2}h+10)$

$= \frac{1}{2}h^2 + 18h + 160$

$\Delta V = (\frac{1}{2}h^2 + 18h + 160)\Delta h$

$V = \lim_{\Delta h \to 0}\sum_{h=0}^{4}(\frac{1}{2}h^2 + 18h + 160)\Delta h$

$= \int_{0}^{4}(\frac{1}{2}h^2 + 18h + 160)\,dh$

$= \left[\frac{1}{6}h^3 + 9h^2 + 160h\right]_0^4$

$= \frac{32}{3} + 144 + 640$

$= \underline{794\frac{2}{3} \text{ units}^3}$ (4)