GIRRAWEEN HIGH SCHOOL
MATHEMATICS

Year 12 Extension 2 Task ~~3~~ 4 ? Thursday 9th June 2005

Instructions: a) Write all your answers on your own paper.
b) Show all necessary working.
c) Marks may be deducted for careless or badly arranged work.

Time Allowed: 90 minutes

Marks

Question 1 *(25 marks)*

Find the following integrals:

(i) $\int \frac{1}{x\log x}dx$ *3*

(ii) $\int \frac{x^2}{1+x}dx$ *3*

(iii) $\int \sin^4 x\cos^3 x dx$ *4*

(iv) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$ *4*

(v) $\int x\tan^{-1} x dx$ *4*

(vi) $\int \frac{3x+1}{x^2+2x+2}dx$ *3*

(vii) $\int \frac{x^2-x-21}{(2x-1)(x^2+4)}dx$ *4*

Question 2 *(13 marks)*

a) (i) If $I_n = \int \tan^n x dx$ for $n \geq 0$ show that $I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$ for $n \geq 2$ 3

(ii) Evaluate $\int_0^{\frac{\pi}{4}} \tan^7 x dx$ 3

b) (i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ 2

(ii) Consider $f(x) = \frac{1}{1+\tan x}$ where $0 \leq x \leq \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 0$ 2

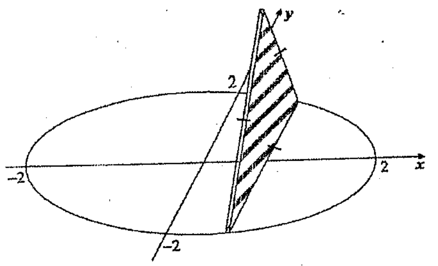
Show that $f(x)+f\left(\frac{\pi}{2}-x\right) = 1$

(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x}dx$ 3

Question 3 *(22 marks)* *Marks*

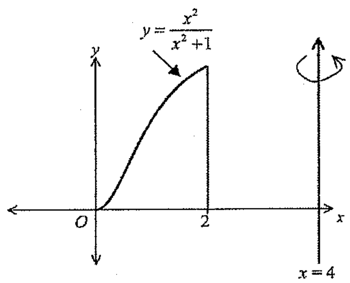
a) The region between the curve $y = \sin x$, the line $y = 1$ and the y axis is rotated about the line $y = 1$. Using a slicing technique find the volume formed. 5

b) 5



The diagram shows a cross-sectional slice of a solid whose base is the region enclosed by the circle $x^2 + y^2 = 4$. Each such cross-section of the solid is an equilateral triangle. Find the volume of the solid.

c) The region bounded by the curve $y = \dfrac{x^2}{x^2+1}$, the x axis and $0 \le x \le 2$, is rotated about the line $x = 4$ to form a solid.



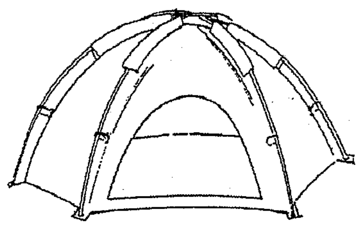
(i) Using the method of cylindrical shells, explain why the volume ΔV of a typical shell distant x units from the origin and with thickness Δx is given by 3

$$\Delta V = 2\pi(4 - x)\left(1 - \frac{1}{1+x^2}\right)\Delta x$$

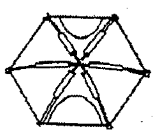
(ii) Hence, find the total volume of the solid formed. 3

d) (i) Show that the area of a regular hexagon of side s is given by $A = \dfrac{3\sqrt{3}s^2}{2}$ 2

(ii) The diagrams below illustrate a dome tent. When erected, the base is a regular hexagon which measures 2 metres from corner to adjacent corner. 4
Flexible exterior poles extend between opposite corners in semi-circle arcs to support the tent.
By taking slices parallel to the base of the tent find the volume enclosed by the tent.



Top View



Year 12 Extension 2 Task 3 2005 Solutions

Question 1 (25)

(i) $\int \frac{1}{x\log x}\,dx$ $\quad u = \log x$, $du = \frac{dx}{x}$

$= \int \frac{du}{u}$

$= \log u + c$

$= \underline{\log\log x + c}$ (3)

(ii) $\int \frac{x^2}{1+x}\,dx$

$$\begin{array}{r} x-1 \\ x+1\,\overline{)\,x^2+0x+0} \\ x^2+x \\ \hline -x+0 \\ -x-1 \\ \hline 1 \end{array}$$

$= \int \left[x - 1 + \frac{1}{1+x}\right]dx$

$= \underline{\frac{1}{2}x^2 - x + \log(1+x) + c}$ (3)

(iii) $\int \sin^4 x\cos^3 x\,dx$ $\quad u = \sin x$, $du = \cos x\,dx$

$= \int \sin^4 x(1-\sin^2 x)\cos x\,dx$

$= \int (u^4 - u^6)\,du$

$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + c$

$= \underline{\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c}$ (4)

(iv) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$ $\quad t = \tan\frac{x}{2}$, $dt = \frac{2dt}{1+t^2}$

$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$

$= \int_0^1 \frac{2dt}{1+t^2+2t+1-t^2}$

$= \int_0^1 \frac{dt}{t+1}$

$= \left[\log(t+1)\right]_0^1$

$= \underline{\log 2}$ (4)

(v) $\int x\tan^{-1}x\,dx$ $\quad u = \tan^{-1}x$, $v = \frac{1}{2}x^2$, $du = \frac{dx}{1+x^2}$, $dv = x\,dx$

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\left[1 - \frac{1}{1+x^2}\right]dx$ (4)

$= \underline{\frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + c}$

(vi) $\int \frac{3x+1}{x^2+2x+2}\,dx$

$= \frac{3}{2}\int \frac{2x+2}{x^2+2x+2}dx - 2\int \frac{dx}{(x+1)^2+1}$ (4)

$= \underline{\frac{3}{2}\log(x^2+2x+2) - 2\tan^{-1}(x+1) + c}$

(vii) $\int \frac{x^2-x-21}{(2x-1)(x^2+4)}\,dx$

$\frac{A}{(2x-1)} + \frac{Bx+C}{(x^2+4)} = \frac{x^2-x-21}{(2x-1)(x^2+4)}$

$A(x^2+4) + (Bx+C)(2x-1) = x^2 - x - 21$

$x = \frac{1}{2}$: $\quad \frac{17}{4}A = -\frac{85}{4}$, $\quad A = -5$

$x = 0$: $\quad 4A - C = -21$, $\quad -20 - C = -21$, $\quad -C = -1$, $\quad C = 1$

$x = 1$: $\quad 5A + B + C = -21$, $\quad -25 + B + 1 = -21$, $\quad B = 3$

$\int \frac{x^2-x-21}{(2x-1)(x^2+4)}\,dx$

$= \int\left[\frac{-5}{2x-1} + \frac{3x+1}{x^2+4}\right]dx$

$= \int\left[\frac{-5}{2x-1} + \frac{3}{2}\cdot\frac{2x}{x^2+4} + \frac{1}{x^2+4}\right]dx$

$= \underline{-\frac{5}{2}\log(2x-1) + \frac{3}{2}\log(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + c}$

(4)

Question 2 (13)

a) $I_n = \int \tan^n x\,dx$

$= \int \tan^{n-2}x\cdot(\sec^2 x - 1)\,dx$

$= \int \tan^{n-2}x\sec^2 x\,dx - \int \tan^{n-2}x\,dx$

$u = \tan x$, $du = \sec^2 x\,dx$

$= \int u^{n-2}\,du - I_{n-2}$

$= \frac{1}{n-1}u^{n-1} - I_{n-2}$

$= \underline{\frac{1}{n-1}\tan^{n-1}x - I_{n-2}}$ (3)

(ii) $\int_0^{\frac{\pi}{4}} \tan^7 x\,dx$

$= \left[\frac{1}{6}\tan^6 x\right]_0^{\frac{\pi}{4}} - I_5$

$= \frac{1}{6} - \left[\frac{1}{4}\tan^4 x\right]_0^{\frac{\pi}{4}} + I_3$

$= \frac{1}{6} - \frac{1}{4} + \left[\frac{1}{2}\tan^2 x\right]_0^{\frac{\pi}{4}} - I_1$

$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \int_0^{\frac{\pi}{4}} \tan x\,dx$

$= \frac{5}{12} + \left[\log\cos x\right]_0^{\frac{\pi}{4}}$

$= \underline{\frac{5}{12} + \log\left(\frac{1}{\sqrt{2}}\right)}$ (3)

b) (i) $\int_0^a f(x)\,dx$ $\quad u = a - x$, $du = -dx$, $x = 0, u = a$; $x = a, u = 0$

$= -\int_a^0 f(a-u)\,du$

$= \int_0^a f(a-u)\,du$

$= \underline{\int_0^a f(a-x)\,dx}$ (2)

(ii) $f(x) + f(\frac{\pi}{2} - x)$

$= \frac{1}{1+\tan x} + \frac{1}{1+\tan(\frac{\pi}{2}-x)}$

$= \frac{1}{1+\tan x} + \frac{1}{1+\cot x}$

$= \frac{1}{1+\tan x} + \frac{1}{1+\frac{1}{\tan x}}$

$= \frac{1}{1+\tan x} + \frac{\tan x}{\tan x + 1}$

$= \frac{1+\tan x}{1+\tan x}$

$= \underline{1}$ (2)

(iii) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x}\,dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan(\frac{\pi}{2}-x)}\,dx$

$= \int_0^{\frac{\pi}{2}} \left[1 - \frac{1}{1+\tan x}\right]dx$

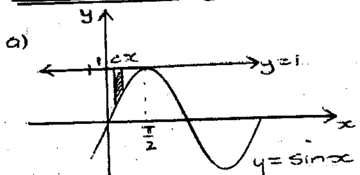
$\therefore 2\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x}\,dx = \int_0^{\frac{\pi}{2}} dx$

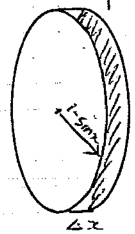
$\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x}\,dx = \frac{1}{2}\left[x\right]_0^{\frac{\pi}{2}}$

$= \underline{\frac{\pi}{4}}$ (3)

Question 3 (25)

a)





$A(x) = \pi(1-\sin x)^2$

$\Delta V = \pi(1-\sin x)^2 \Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=0}^{\frac{\pi}{2}} \pi(1-\sin x)^2 \Delta x$

$= \pi \int_0^{\frac{\pi}{2}} (1 - 2\sin x + \sin^2 x)\,dx$

$= \pi \int_0^{\frac{\pi}{2}} \left(1 - 2\sin x + \tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) dx$

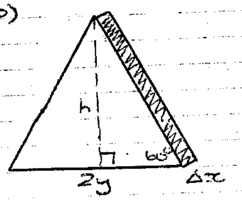
$= \pi \left[\tfrac{3}{2}x + 2\cos x - \tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$

$= \pi\left(\frac{3\pi}{4} + 0 - 0 - 0 - 2 + 0\right)$

$= \pi\left(\frac{3\pi}{4} - 2\right)$

$= \dfrac{3\pi^2 - 8\pi}{4}$ units^3 (5)

b)



$\frac{h}{y} = \tan 60^\circ$

$h = y\tan 60^\circ$

$h = \sqrt{3}\,y$

$A(x) = \tfrac{1}{2}(2y)(\sqrt{3}y)$

$= \sqrt{3}\,y^2$

$= \sqrt{3}(4 - x^2)$

$\Delta V = \sqrt{3}(4 - x^2)\Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=-2}^{2} \sqrt{3}(4 - x^2)\Delta x$

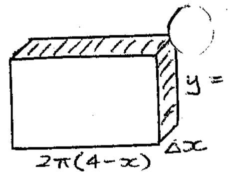
$= 2\sqrt{3}\int_0^2 (4 - x^2)\,dx$

$= 2\sqrt{3}\left[4x - \tfrac{1}{3}x^3\right]_0^2$

$= 2\sqrt{3}\left(8 - \tfrac{8}{3}\right)$

$= \dfrac{32\sqrt{3}}{3}$ units^3 (5)

c)



$y = \dfrac{x^2}{x^2+1}$

$A(x) = 2\pi(4-x)\left(\dfrac{x^2}{x^2+1}\right)$

$= 2\pi(4-x)\left(1 - \dfrac{1}{x^2+1}\right)$ (3)

$\Delta V = 2\pi(4-x)\left(1 - \dfrac{1}{x^2+1}\right)\Delta x$

(ii) $V = \lim_{\Delta x \to 0} \sum_{x=0}^{2} 2\pi(4-x)\left(1 - \dfrac{1}{x^2+1}\right)\Delta x$

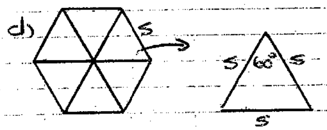
$V = 2\pi\int_0^2 \left[4 - x - \dfrac{4}{x^2+1} + \dfrac{x}{x^2+1}\right] dx$

$= 2\pi\left[4x - \tfrac{1}{2}x^2 - 4\tan^{-1}x + \tfrac{1}{2}\log(x^2+1)\right]_0^2$

$= 2\pi\left(8 - 2 - 4\tan^{-1}2 + \tfrac{1}{2}\log 5 - 0\right)$

$= 2\pi\left(6 - 4\tan^{-1}2 + \tfrac{1}{2}\log 5\right)$ units^3 (3)

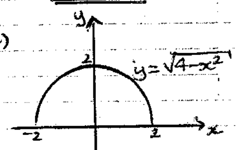
d)

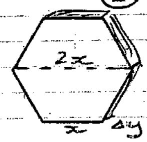


$A = 6 \times \tfrac{1}{2}(s)(s)\sin 60^\circ$

$= \dfrac{3\sqrt{3}s^2}{2}$ (2)

(ii)





$A(y) = \dfrac{3\sqrt{3}}{2}x^2$

$= \dfrac{3\sqrt{3}}{2}(4 - y^2)$

$\Delta V = \dfrac{3\sqrt{3}}{2}(4 - y^2)\Delta y$

$V = \lim_{\Delta y \to 0} \sum_{y=0}^{2} \dfrac{3\sqrt{3}}{2}(4 - y^2)\Delta y$

$= \dfrac{3\sqrt{3}}{2}\int_0^2 (4 - y^2)\,dy$

$= \dfrac{3\sqrt{3}}{2}\left[4y - \tfrac{1}{3}y^3\right]_0^2$

$= \dfrac{3\sqrt{3}}{2}\left(8 - \tfrac{8}{3}\right)$

$= 8\sqrt{3}$ m^3 (5)