GIRRAWEEN HIGH SCHOOL

YEAR 12 - TASK 3

2006

MATHEMATICS

Extension 2

Time allowed – 90 minutes

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a *new* sheet of paper.

Question 1 (28 marks)

Marks

(a) $\int \sin^4 x dx$ **4**

(b) $\int x^2 \cos x dx$ **4**

(c) $\int \tan^{-1} x dx$ **4**

(d) $\int \cos^2 x \cot x dx$ **4**

(e) $\int \frac{1}{2+\cos x} dx$ **4**

(f) $\int \frac{5+3x}{1-9x^2} dx$ **4**

(g) $\int \frac{1}{\sqrt{5+4x-x^2}} dx$ **4**

Question 2 (14 marks)

(a) Show that

(i) $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$ **2**

(ii) and hence solve $\int x^3 e^x dx$ **3**

(b) Calculate the area bounded by the curve $y = \frac{1}{x^2(x-3)}$,

and the x-axis and the ordinates $x = 4$ and $x = 6$. **5**

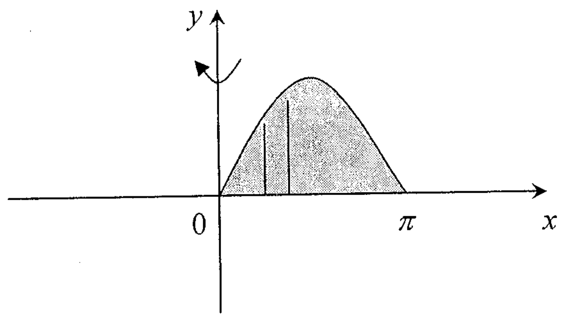
(c) Calculate the area of the region bounded by the curve $y = xe^{-x}$, the x-axis and the line $x = 1$ **4**

Question 3 (20 marks)

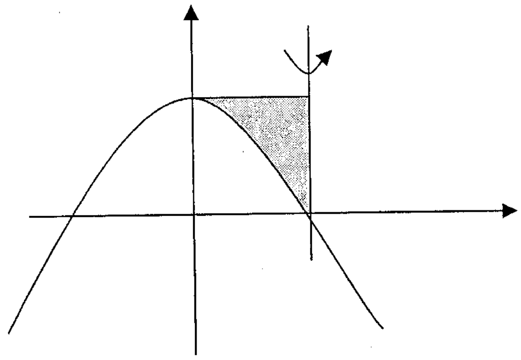
Marks

(a) Find the volume generated when the area in the first quadrant bounded by $y = x$ and $y = x^3$ is revolved about the x-axis. **4**

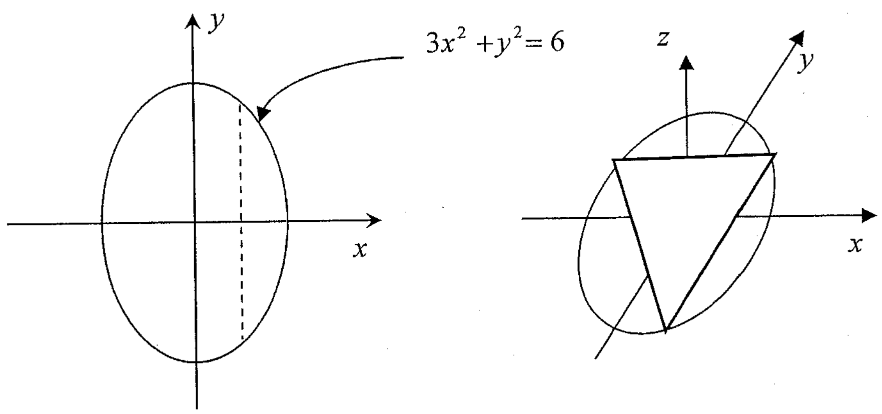
(b) Find the volume generated when the area bounded by $y = \sin x$, $y = 0$, between $x = 0$ and $x = \pi$ is revolved about the y-axis, using cylindrical shells. **6**



(c) The region bounded by $y = 1 - x^2$ and the lines $x = 1, y = 1$ is rotated about $x = 1$. Find the volume generated. **5**



(d) The base of a solid is the region enclosed by the ellipse $3x^2 + y^2 = 6$. Find the volume of the solid if all the plane sections perpendicular to the x-axis are equilateral triangles. **5**



Mathematics (Ext 2) - 2006 {Girraween}
TASK 3 (HSC) SOLUTIONS

Q1 a) $\int \sin^4 x\, dx$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\therefore \sin^4 x = \frac{1}{4}(1-\cos 2x)^2$

$= \frac{1}{4}\int (1-\cos 2x)^2\, dx$

$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x\, dx$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\therefore \cos^2 2x = \frac{1}{2}(1+\cos 4x)$

$= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\, dx$

$= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right] + C$ (4)

b) $\int x^2 \cos x\, dx$

$u = x^2 \quad v' = \cos x$
$u' = 2x \quad v = \sin x$

$= x^2 \sin x - \int 2x \sin x\, dx$

$u = 2x \quad v' = \sin x$
$u' = 2 \quad v = -\cos x$

$= x^2 \sin x - \left[2x.-\cos x - \int 2.-\cos x\, dx\right]$

$= x^2 \sin x + 2x\cos x - 2\sin x + C$ (4)

c) $\int \tan^{-1} x\, dx$

$u = \tan^{-1} x \quad v' = 1$
$u' = \frac{1}{1+x^2} \quad v = x$

$= x\tan^{-1}x - \int \frac{x}{1+x^2}\, dx$

$= x\tan^{-1}x - \frac{1}{2}\log_e(1+x^2) + C$ (4)

d) $\int \cos^2 x \cot x\, dx$

$= \int \frac{\cos^3 x}{\sin x}\, dx$

$= \int \frac{(1-\sin^2 x)}{\sin x}.\cos x\, dx$

Let $u = \sin x$
$du = \cos x\, dx$

$= \int \frac{1-u^2}{u}.\, du$

$= \int \frac{1}{u} - u.\, du$

$= \log_e u - \frac{u^2}{2} + c$

$= \log_e(\sin x) - \frac{1}{2}\sin^2 x + C$ (4)

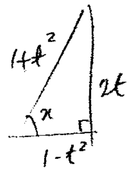
e) $\int \frac{1}{2+\cos x}\, dx$

$t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$dt = \frac{1}{2}(1+\tan^2\frac{x}{2})d$

$dx = \frac{2dt}{1+t^2}$



$= \int \frac{1}{2+\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2}$

$= \int \frac{2\, dt}{2(1+t^2)+1-t^2}$

$= \int \frac{2\, dt}{2+2t^2+1-t^2}$

$= 2\int \frac{dt}{3+t^2}$

$= 2.\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + C$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + C$ (4)

f) $\int \frac{5+3x}{1-9x^2}\, dx$

$= \int \frac{5+3x\, dx}{(1-3x)(1+3x)}$

$\frac{5+3x}{(1-3x)(1+3x)} = \frac{a}{(1-3x)} + \frac{b}{(1+3x}$

$5+3x = a(1+3x) + b(1-3$

$= \int \frac{3}{1-3x} + \frac{2}{1+3x}\, dx$

when $x = \frac{1}{3}$ $\quad 6 = 2a$

$a = 3$

$= -\log_e(1-3x) + \frac{2}{3}\log_e(1+3x) + C$

$= \frac{2}{3}\log_e(1+3x) - \log_e(1-3x) + C$

when $x = -\frac{1}{3}$ $\quad 4 = 2b$

$b = 2$ (4)

g) $\int \frac{1}{\sqrt{5+4x-x^2}}\, dx$

$5+4x-x^2$
$= 9-(x^2-4x+4)$
$= 9-(x-2)^2$

$= \int \frac{1\, dx}{\sqrt{9-(x-2)^2}}$

Let; $u = (x-2)$
$du = dx$

$= \int \frac{du}{\sqrt{3^2-u^2}}$

$= \sin^{-1}\left(\frac{u}{3}\right) + c$

$= \sin^{-1}\left(\frac{x-2}{3}\right) + c$ (4)

Q2

a) i) $\int x^n e^x \, dx$

$u = x^n \quad v' = e^x$

$u' = nx^{n-1} \quad v = e^x$

$= x^n e^x - \int n x^{n-1} . e^x \, dx$

$= x^n e^x - n \int x^{n-1} e^x . \, dx$ (2)

$I_n = x^n e^x - n \, I_{n-1}$

ii) $I_3 = \int x^3 e^x \, dx$

$I_3 = x^3 e^x - 3 I_2 \quad ; \quad I_2 = x^2 e^x - 2 I_1$

$= x^3 e^x - 3[x^2 e^x - 2 I_1] \quad ; \quad I_1 = x e^x - 1 I_0$

$= x^3 e^x - 3x^2 e^x + 6[x e^x - I_0] \quad I_0 = \int x^0 e^x \, dx$

$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \quad \therefore I_0 = \int e^x \, dx$

(3) $I_0 = e^x$

b) $y = \frac{1}{x^2(x-3)} = \frac{ax+b}{x^2} + \frac{c}{(x-3)}$

$1 = (ax+b)(x-3) + cx^2$

$A = \int_4^6 \frac{1}{x^2(x-3)} \, dx$

when $x=3$: $1 = 9c$, $c = \frac{1}{9}$

$= \int_4^6 \frac{-\frac{1}{9}x - \frac{1}{3}}{x^2} + \frac{\frac{1}{9}}{x-3} \, dx$

$x=0$: $1 = -3b$, $b = -\frac{1}{3}$

$= \frac{1}{9} \int_4^6 \frac{-x-3}{x^2} + \frac{1}{x-3} \, dx$

$x=1$, $c = \frac{1}{9}$, $b = -\frac{1}{3}$:

$1 = (a+b)(-2) + c$

$1 = -2a - 2b + c$

$= \frac{1}{9} \int_4^6 -\frac{1}{x} - 3x^{-2} + \frac{1}{x-3} \, dx$

$1 = -2a + \frac{2}{3} + \frac{1}{9}$

$= \frac{1}{9} \left[-\log_e(x) + \frac{3}{x} + \log_e(x-3) \right]_4^6$

$9 = -18a + 6 + 1$

$-18a = 2$

$a = -\frac{1}{9}$

$= \frac{1}{9} \left[\log_e\left(\frac{x-3}{x}\right) + \frac{3}{x} \right]_4^6$

$= \frac{1}{9} \left[\left(\log_e \frac{1}{2} + \frac{1}{2}\right) - \left(\log_e \frac{1}{4} + \frac{3}{4}\right) \right]$

$= \frac{1}{9} \left[\log_e 2 - \frac{1}{4} \right]$ (5)

c) $y = x e^{-x}$

when $y=0$, $x=0$

$A = \int_0^1 x e^{-x} \, dx$

$u = x \quad v' = e^{-x}$

$u' = 1 \quad v = -e^{-x}$

$= -x e^{-x} - \int -e^{-x} \, dx$

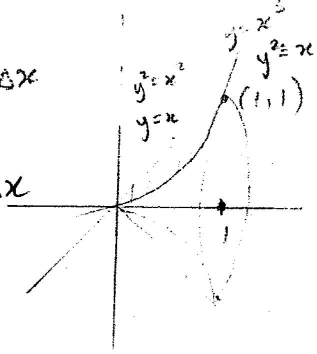
$= \left[-x e^{-x} - e^{-x} \right]_0^1$

$= (-e^{-1} - e^{-1}) - (0 - 1)$

$= -\frac{1}{e} - \frac{1}{e} + 1$

$= 1 - \frac{2}{e}$ (4)

(Q3) $V = \lim_{\Delta x \to 0} \sum_{x=a}^{b} \pi (R^2 - r^2).\Delta x$



$V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} \pi (x^2 - x^6).\Delta x$

$= \pi \int_0^1 x^2 - x^6 \, dx$

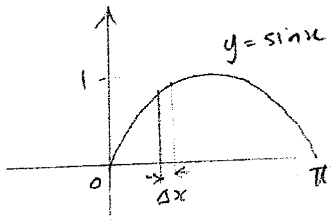
$= \pi \left[\frac{x^3}{3} - \frac{x^7}{7} \right]_0^1$

$= \pi \left[\frac{1}{3} - \frac{1}{7} - (0 - 0) \right]$

$= \frac{4\pi}{21}$

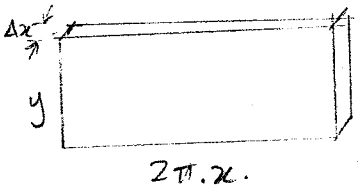
(4)

b) $V = \lim_{\Delta x \to 0} \sum_{x=0}^{\pi} 2\pi x.y.\Delta x$



$V = 2\pi \int_0^{\pi} x.\sin x.dx$

$= 2\pi \left[-x.\cos x - \int -\cos x \, dx \right]_0^{\pi}$



$= 2\pi \left[-x.\cos x + \sin x \right]_0^{\pi}$

$= 2\pi \left[(\pi + 0) - (0 + 0) \right]$ $\quad u = x \quad v' = \sin x$

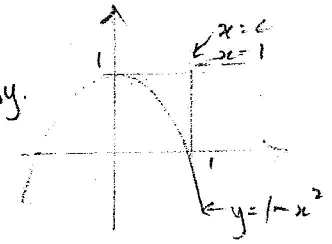
$\quad u' = 1 \quad v = -\cos x$

$= 2\pi^2$

(6)

$V = \lim_{\Delta y \to 0} \sum_{x=a}^{b} (c - x)^2 \Delta y$

c) $V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} (1 - \sqrt{1-y})^2.\Delta y$



$= \pi \int_0^1 (1 - \sqrt{1-y})^2 dy$

$x^2 = 1 - y$

$x = \sqrt{1-y}$

$= \pi \int_0^1 1 - 2\sqrt{1-y} + (1-y) \, dy$

$= \pi \int_0^1 2 - y - 2\sqrt{1-y} \, dy$

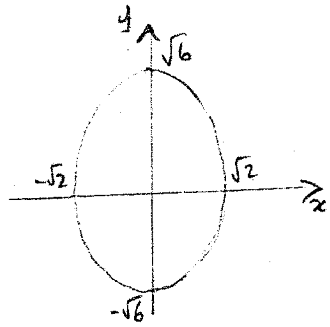
$= \pi \left[2y - \frac{y^2}{2} - \frac{2(1-y)^{3/2}}{3/2(-1)} \right]_0^1$

$= \pi \left[\left(2 - \frac{1}{2} + 0\right) - \left(0 - 0 + \frac{4}{3}\right) \right]$

$= \frac{\pi}{6}$

(5)

d) $3x^2 + y^2 = 6$

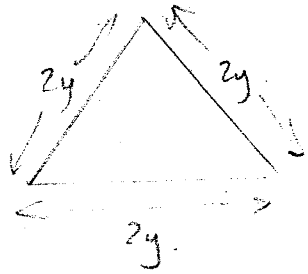


$\frac{x^2}{2} + \frac{y^2}{6} = 1$

$\therefore a = \sqrt{2} \quad b = \sqrt{6}.$

$V = \lim_{\Delta x \to 0} \sum_{x=-\sqrt{2}}^{\sqrt{2}} A(x).\Delta x.$

$V = \lim_{\Delta x \to 0} \sum_{x=-\sqrt{2}}^{\sqrt{2}} \sqrt{3}(6 - 3x^2).\Delta x$



$V = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{3}(6 - 3x^2) dx.$

$= \sqrt{3} \int_{-\sqrt{2}}^{\sqrt{2}} 6 - 3x^2 dx$

Area$_{(x)}$ $= \frac{1}{2}(2y)(2y)\sin$

$= 2y^2.\left(\frac{\sqrt{3}}{2}\right)$

$= \sqrt{3}y^2$

$= 3\sqrt{3} \int_{-\sqrt{2}}^{\sqrt{2}} 2 - x^2 dx$ ← even fn.

$= 6\sqrt{3} \int_0^{\sqrt{2}} 2 - x^2 dx$

Since $\left[y^2 = 6 - 3x^2 \right]$

$\therefore$ Area $= \sqrt{3}(6 - 3$

$= 6\sqrt{3} \left[2x - \frac{x^3}{3} \right]_0^{\sqrt{2}}$

$= 6\sqrt{3} \left[\left(2\sqrt{2} - \frac{2\sqrt{2}}{3}\right) - (0) \right]$

$= 6\sqrt{3} \left(\frac{6\sqrt{2} - 2\sqrt{2}}{3} \right)$

$= 2\sqrt{3}(4\sqrt{2})$

$= 8\sqrt{6}.$

(5)